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PREFACE

This book is intended to serve as a text-book on the theory of analytic functions and covers the syllabus of the Physico-Mathematical faculties of the universities. Numerous examples illustrating the general propositions and methods have also been included, duly supplemented with problems and details. For the readers who would like to obtain a more advanced knowledge in this field, the author has cited a number of references at the end of the book.

While preparing the present text the author has drawn extensively upon his earlier title, "*Theory of Analytic Functions*".

A. I. Markushevich

CONTENTS

Preface v

INTRODUCTION 1

1. The scope of the theory of analytic functions 1
2. Analytic functions of a complex variable 2

1. COMPLEX NUMBERS AND THEIR GEOMETRICAL REPRESENTATION 5

1. Geometrical representation of complex numbers on a plane . 5
2. Operations with complex numbers 7
3. The limit of a sequence 9
4. The point at infinity and the stereographic projection . . . 10
5. Point sets in a plane 13

2. FUNCTIONS OF A COMPLEX VARIABLE. DERIVATIVE AND ITS GEOMETRICAL AND HYDROMECHANICAL INTERPRETATION 17

1. Functions of a complex variable 17
2. Limit of a function at a point 17
3. Continuity 19
4. Continuous curve 20
5. Derivative and differential 24
6. Rules of differentiation 26
7. Necessary and sufficient conditions for differentiability at an interior point of a domain 27
8. Geometrical interpretation of the argument of a derivative . 33
9. Geometrical interpretation of the modulus of a derivative . 35
10. Example: linear and bilinear functions 36
11. The angle formed at the point at infinity 37
12. Harmonic and conjugate harmonic functions 39
13. Hydromechanical interpretation of an analytic function . . 43
14. Examples 49

3. ELEMENTARY ANALYTIC FUNCTIONS AND THE CORRESPONDING CONFORMAL MAPPINGS 52

1. The polynomial 52
2. Points at which the conformality of the mapping fails . . . 53
3. Mapping of the form $w = (z-a)^n$ 54
4. Group properties of the bilinear transformations 58
5. The circle-preserving property 61
6. Invariance of the cross ratio 64
7. Mapping of domains bounded by straight lines or circles . . 70
8. Symmetry and its preservation 72
9. Examples 76
10. The Joukowski function 79
11. Definition of an exponential function 84
12. Mapping by an exponential function 86
13. Trigonometric functions 91
14. Geometrical behaviour 96
15. Continuation 100
16. One-valued branches of the many-valued functions 102
17. The function $w = \sqrt[n]{z}$ 104
18. The function $w = \sqrt[n]{P(z)}$ 109
19. Logarithm 114
20. The general power function and exponential function . . . 119
21. Inverse trigonometric functions 125

4. COMPLEX TERM SERIES. POWER SERIES 130

1. Convergent and divergent series 130
2. The Cauchy-Hadamard theorem 132
3. Analyticity of the sum of a power series 135
4. Uniform convergence 138

5. INTEGRATION OF A FUNCTION OF A COMPLEX VARIABLE 140

1. Integral of a function of a complex variable 140
2. The properties of integrals 143
3. Reduction to the calculation of an ordinary integral . . . 144
4. The integral theorem of Cauchy 146
5. The integral theorem of Cauchy : proof (continued) . . . 150
6. Application to the calculation of definite integrals 152
7. Integral and the primitive 162
8. Generalization of the integral theorem of Cauchy when the function is not analytic on the contour of integration . . . 165
9. The theorem on a composite contour 166
10. Integral as a function of a point in a multiply-connected domain 169

6. THE CAUCHY INTEGRAL FORMULA AND ITS CONSEQUENCES 173

1. The Cauchy integral formula 173
2. Decomposition of an analytic function into a power series. The theorem of Liouville 175
3. Infinite differentiability of analytic and harmonic functions . 178
4. The theorem of Morera 182
5. Weierstrass's theorem on the uniformly convergent series of analytic functions 183
6. The uniqueness theorem 187
7. A-points and, in particular, the zeros 190
8. Series of power series 191
9. Substitution of one series in another 194
10. Division of the power series 198
11. Expansion of the functions cot z, tan z, cosec z and sec z into power series 204
12. Expansion of harmonic functions into series. The Poisson integral and the Schwartz formula 208

7. THE LAURENT SERIES. ISOLATED SINGULAR POINTS OF A SIMPLE CHARACTER. ENTIRE AND MEROMORPHIC FUNCTIONS 215

1. The Laurent series 215
2. Laurent's theorem 219
3. Isolated singular points of a simple character 222
4. The theorem of Sokhotskii 227
5. Singularities of the derivatives and of the rational combinations of analytic functions 231
6. The case of the point at infinity 235
7. Entire and meromorphic functions 236
8. Expansion of an entire function into a product 241
9. The order and the type of an entire function 248

8. RESIDUES AND THEIR APPLICATIONS. THE PRINCIPLE OF THE ARGUMENT 251

1. The theorem of residues and its application to the evaluation of definite integral 251
2. The principle of the argument and its consequences 258
3. Residue at the point at infinity 265
4. Application of the theorem of residues to the expansion of meromorphic functions into partial fractions 267

5. Expansion of sec z, cot z, cosec z and tan z into partial fractions 273

9. ANALYTIC CONTINUATION. THE CONCEPT OF A RIEMANN SURFACE. SINGULARITIES 283

1. Analytic continuation 283
2. Direct analytic continuation 285
3. The construction of an analytic function from its elements . 287
4. Construction of a Riemann surface 288
5. The Riemann-Schwartz principle of symmetry 291
6. Singularities on the boundary of the circle of convergence of a power series 295
7. Criterion for locating singularities 300
8. Determination of the radius of convergence of a power series from a known distribution of singularities of the function . 304
9. Isolated singularities of a multiple character 309

10. MAPPINGS BY ANALYTIC FUNCTIONS. THE CONCEPT OF ELLIPTIC FUNCTIONS. THE CHRISTOFFEL-SCHWARTZ FORMULA 315

1. Mapping of a domain by an analytic function 315
2. The principle of the maximum modulus and the Schwartz lemma 316
3. Local criterion of single-sheetedness 318
4. Inversion of an analytic function 320
5. The extension of the concept of single-sheetedness to functions having poles 324
6. The concept of the Riemann theorem. The uniqueness of mapping 326
7. The concept of the correspondence between boundaries. The converse theorem 327
8. Mapping of the upper half-plane by the elliptic integral . . 334
9. The concept of the Jacobian elliptic function sn w 339
10. The Christoffel-Schwartz integral 344
11. Flow past a circular cylinder (without circulation) 351
12. Hydromechanical interpretation of the simplest singularities. 353
13. The general solution of the problem of flow past a circular cylinder 358
14. Determination of the lift of an aerofoil 363

References 369

Index 372

INTRODUCTION

1. THE SCOPE OF THE THEORY OF ANALYTIC FUNCTIONS

The subject discussed in this book actually falls under two headings—the theory of analytic functions and the theory of functions of a complex variable. Whereas each of these terms emphasizes only one aspect of the matter in hand, we shall study here the analytic functions of a complex variable.

The function $f(x)$ of the real variable x, defined in some interval (a, b), finite or infinite, is regarded to be analytic in this interval if, in the neighbourhood of any point x_0 in it, this function can be represented in the form of a series expansion in nonnegative integral powers of $x - x_0$:

$$f(x) = A_0 + A_1 (x-x_0) + A_2 (x-x_0)^2 + \ldots + A_n (x-x_0)^n + \ldots .$$

Any polynomial in x, the functions e^x, $\sin x$ and $\cos x$ are all analytic on the whole of the real axis. Any rational function and the functions $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ are analytic in intervals not containing the points at which the corresponding function is left undefined (and goes to infinity); the function $\ln x$ is analytic in the interval $(0, \infty)$. All these statements can be readily proved. For example, if $x_0 > 0$, then

$$\ln x = \ln x_0 + \ln\left(1+\frac{x-x_0}{x_0}\right) = \ln x_0 + \sum_1^\infty (-1)^{n-1} \frac{(x-x_0)^n}{nx_0^n}$$

$$\text{when} \quad |x - x_0| < |x_0|.$$

The sum, the difference, the product and the quotient of analytic functions (in an interval in which the denominator does not vanish) are themselves analytic functions. The derivative and the integral of an analytic function are also analytic. Subject to certain reservations, the

following propositions hold : (*a*) a function inverse to an analytic function is also analytic, (*b*) if $A_j(x)$ ($j=0, 1, \ldots, n$) are analytic functions, the function $f(x)$ defined by the equation

$$A_0(x) + A_1(x) f(x) + \ldots + A_n(x) [f(x)]^n = 0,$$

or by the equation

$$A_0(x) + A_1(x) \frac{df(x)}{dx} + \ldots + A_n(x) \frac{d^n f(x)}{dx^n} = 0,$$

is also analytic.

From these propositions we can see that the most important functions, to which problems of mathematical analysis, geometry, mechanics and physics lead, are all analytic. As a matter of fact, not only the elementary functions mentioned above but also such functions as the gamma functions, the cylindrical (Bessel) functions, the elliptic functions and several other classes of functions are found to be analytic in the corresponding intervals. This is the reason why analytic functions play such a vital role in mathematics and in its applications; we are, therefore, justified in establishing a separate mathematical discipline for the general study of analytic functions.

2. ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

While studying the simplest analytic function—the polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \quad (a_n \neq 0),$$

it has been found expedient to look upon it as a function of a complex variable.

As a matter of fact, only such an approach reveals that this function takes up any value, particularly the value zero, for n values of x (some of which might, of course, coincide). Hence, in sequel, a number of fundamental results follow from this, for example, a polynomial may be represented in the form of a product of linear factors

$$f(x) = a_n (x-x_1)(x-x_2) \ldots (x-x_n),$$

and others, connected with this proposition.

It is natural that while studying polynomials as functions of a complex variable, arbitrary complex numbers can be taken as the values of their coefficients. Similarly, while studying the most general analytic functions of a complex variable $z = x + iy$, we use the power series $\sum_0^\infty A_n (z-z_0)^n$ in which the coefficients $A_0, A_1, \ldots, A_n, \ldots$, as well as the

number z_0 are complex numbers. The function $f(z)$ is said to be *analytic at a certain set of points of the complex plane (representing the complex numbers)*, if in the neighbourhood of any of these points this function can be represented in the form of a power series

$$f(z) = \sum_0^\infty A_n (z-z_0)^n.$$

It must be made clear that, in general, the basic concepts of mathematical analysis, including the concepts of the derivative $f'(z)$ and the integral $\int_L f(z)\,dz$, taken along any plane curve L, can be extended to functions of a complex variable.

The following fundamental facts will be proved during the course of our study. *In order that the function $f(z)$ be analytic in a certain circle in the plane of the complex variable z, it is necessary and sufficient that any one of the following four conditions be satisfied :*

(*a*) *the function $f(z)$ has a derivative $f'(z)$ at every point in the circle* ;

(*b*) *if $f(z) = u(x, y) + iv(x, y)$*, where $u(x, y)$ and $v(x, y)$ *are two real functions of the real variables x and y, then $u(x, y)$ and $v(x, y)$ are twice differentiable functions satisfying the Laplace equation*

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

and are related by the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

(*c*) *the function $f(z)$ is continuous within the circle and its integral along any closed curve, lying within the circle, is equal to zero* ;

(*d*) *in any concentric circle with a lesser radius the function $f(z)$ can be approximated by a polynomial up to any arbitrary degree of accuracy.*

The whole theory of analytic functions of a complex variable can be built upon these propositions. Any of the properties (*a*), (*b*), (*c*) and (*d*) can be made the basis of a definition of the concept of an analytic function of a complex variable. In our study, we shall use the definition based on the property (*a*) and only later will it be shown that this definition is equivalent to the one in which power series are involved.

We note that many of the applications of the theory of analytic

functions in physics and mechanics are based on the property (*b*). Examples of such applications are the so-called plane problems of thermal or electrical equilibrium and those of flow past plane contours set up by liquid or gas motions ; these lead to the Laplace equation, from different solutions of which all analytic functions can be constructed.

1

COMPLEX NUMBERS AND THEIR GEOMETRICAL REPERSENTATION

1. GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS ON A PLANE

The theory of complex numbers is, of course, given in the courses on higher algebra.[1] Here, we only recall the fundamental definitions and the results of this theory and to some extent supplement these in the interest of future presentation.

Every complex number c has the form $a + bi$, where a and b are real numbers and i the so-called imaginary unit ; a is called the *real part* of c while b is called its *imaginary part*. Symbolically, $a = \mathrm{Re}\, c$, $b = \mathrm{Im}\, c$ (Re are the initial letters of the Latin word *realis* meaning real and Im are the initial letters of the word *imaginarius* meaning imaginary). Two complex numbers c' and c'' are equal if, and only if, $\mathrm{Re}\, c' = \mathrm{Re}\, c''$ and $\mathrm{Im}\, c' = \mathrm{Im}\, c''$. If $\mathrm{Im}\, c = 0$, then $c = \mathrm{Re}\, c$ is a real number; if $\mathrm{Im}\, c \neq 0$, c is an imaginary number and if in addition $\mathrm{Re}\, c = 0$ it becomes a pure imaginary number.

For a geometrical representation of complex numbers on a plane we choose a rectangular Cartesian coordinate system and take every point $M(x, y)$ as an image of the complex number $z = x + yi$. The number $x + yi$ is called the *affix* of the point M. This establishes a one-to-one correspondence between the set of all points in the plane and the set of all complex numbers. The set of all real numbers is represented by the axis of the abscissae which, in consequence of this, is called the *real axis*, while the set of all imaginary points—the set of points not lying on the axis of the abscissae and, in particular, the set of pure imaginary numbers,—lies on the axis of the ordinates, the so-called *imaginary axis* ; we observe that one point of the imaginary axis, viz. the origin of

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[1] See A. G. Kurosh, *Course of Higher Algebra*, Physico-mathematical Publishing House, Moscow, 1959, 6th edition ; English translation being published by The Hindustan Publishing Corporation, Delhi-6.

coordinates, represents the real number 0. The plane, the points of which represent complex numbers, is called the *complex plane* (sometimes the Gaussian plane) and also the plane (z), the plane (w), etc., depending upon the letter z, w, . . . denoting the complex number.

The terms 'complex number $x + iy$' and 'the point $x + iy$' are being used synonymously.

For a geometrical representation of the number $z = x + iy$, we use, besides the point (x, y), a vector with projections x and y on the coordinate axes; its origin can be placed at any arbitrary point (Fig. 1). Hence, we can also treat the terms 'complex number' and 'vector' as synonymous. The length $|z|$ of the vector is called the *modulus* of the complex number z and the angle Arg z between the positive direction of the real axis and the vector (assuming that $z \neq 0$) is called the *argument* of z, the latter being defined only to the extent of an integral multiple of 2π. One and only one value α of the argument satisfies the condition $-\pi < \alpha \leqslant \pi$; this value is called the *principal value of the argument* and is denoted by arg z. It is evident that

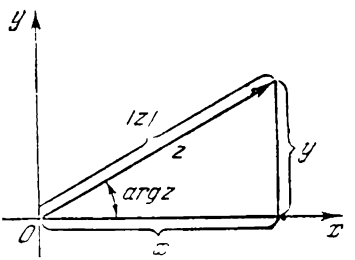


Fig. 1.

$$\operatorname{Arg} z = \arg z + 2k\pi,$$

where k denotes any integer.

We also note the following relations:

If $z = x + iy$, then $|z| = \sqrt{x^2+y^2}$;

$$\arg z = \arctan \frac{y}{x} \quad \text{for} \quad x > 0,$$

$$\arg z = \arctan \frac{y}{x} + \pi \quad \text{for} \quad x < 0 \text{ and } y \geqslant 0,$$

$$\arg z = \arctan \frac{y}{x} - \pi \quad \text{for} \quad x < 0 \text{ and } y < 0.$$

The real and the imaginary parts of z can be expressed in terms of the modulus and the argument in the following manner: $\operatorname{Re} z = |z| \cos \operatorname{Arg} z$ and $\operatorname{Im} z = |z| \sin \operatorname{Arg} z$.

Hence, z itself can be represented in the form $z = |z| (\cos \operatorname{Arg} z + i \sin \operatorname{Arg} z)$, which is called the *trigonometrical form* of z.

The two complex numbers $x + iy$ and $x - iy$ are called (mutually)

conjugate. If one of them is denoted by z, the other is denoted by $\bar{z}$. Evidently, the points z and $\bar{z}$ are symmetrical about the real axis. Hence $|z| = |\bar{z}|$. Moreover, $\arg z = -\arg \bar{z}$ if z is not a real negative number; in the latter case, $\arg z = \arg \bar{z} = \pi$.

2. OPERATIONS WITH COMPLEX NUMBERS

The operations of addition and multiplication of complex numbers are defined by the following equalities:

$$(a_1+ib_1) + (a_2+ib_2) = (a_1+a_2) + i(b_1+b_2),$$

$$(a_1+ib_1)(a_2+ib_2) = (a_1a_2-b_1b_2) + i(a_1b_2+a_2b_1);$$

those of subtraction and division are defined as operations inverse to these. From these definitions follow the important consequences, viz. that addition and multiplication are commutative and associative, multiplication being distributive also; the product of two factors is equal to zero if, and only if, at least one of them is equal to zero; subtraction is always possible, but division is possible only when the divisor is nonzero. All these signify that the complex numbers form a field. We note a particular case of multiplication: if $c = a + ib$, then $c \cdot \bar{c} = a^2 + b^2 = |c|^2$, whence $|c| = \sqrt{c \cdot \bar{c}}$.

The geometrical addition of the numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ follows the law of addition of vectors (Fig. 2,*a*). The difference $c_1 - c_2$ is represented by a vector, with c_2 and c_1 as its initial and

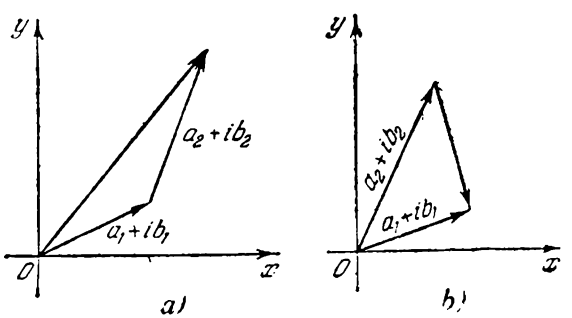


Fig. 2

terminal points, respectively (Fig. 2,*b*). It thus follows that the distance between the two points $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ is equal to the modulus of the difference: $\rho(c_2, c_1) = |c_1 - c_2|$. We further note the important inequalities for the modulus of a sum and a difference:

$$|c_1 + c_2| \leqslant |c_1| + |c_2| \text{ and } |c_1 - c_2| \geqslant ||c_1| - |c_2||:$$

the signs of equality hold only when the vectors c_1 and c_2 are collinear and are directed in the same sense.

The first inequality can be extended to any number of summands:

$$| c_1 + c_2 + \ldots + c_n | \leqslant | c_1 | + \ldots + | c_n | ;$$

in this case the equality can hold only when all the n vectors $c_1, c_2, \ldots, c_n$ are collinear, and directed in the same sense.

For a geometrical interpretation of the process of multiplication, we write c_1 and c_2 in the trigonometrical form:

$$c_1 = | c_1 | (\cos \operatorname{Arg} c_1 + i \sin \operatorname{Arg} c_1),$$
$$c_2 = | c_2 | (\cos \operatorname{Arg} c_2 + i \sin \operatorname{Arg} c_2);$$

then, from the definition of multiplication, we get

$$c = c_1 c_2 = | c_1 | | c_2 | [\cos (\operatorname{Arg} c_1 + \operatorname{Arg} c_2) + i \sin (\operatorname{Arg} c_1 + \operatorname{Arg} c_2)].$$

Hence, it follows that

$$| c_1 \cdot c_2 | = | c_1 | | c_2 | \quad \text{and} \quad \operatorname{Arg} (c_1 \cdot c_2) = \operatorname{Arg} c_1 + \operatorname{Arg} c_2;$$

the last relation should be understood in the sense that by forming all possible sums of the values of $\operatorname{Arg} c_1$ and $\operatorname{Arg} c_2$ we obtain a set of numbers coinciding with the set of values of $\operatorname{Arg} (c_1 \cdot c_2)$. Geometrically, the multiplication of c_1 by c_2 ($c_1 \neq 0$ and $c_2 \neq 0$) signifies that the vector c_1 is magnified $| c_2 |$ times and is rotated (about its origin) through the angle $\operatorname{Arg} c_2$. For the quotient, $c_1 : c_2 = \frac{c_1}{c_2}$ ($c_1 \neq 0$, $c_2 \neq 0$), we obtain the equalities

$$| c_1 : c_2 | = | c_1 | : | c_2 | \quad \text{and} \quad \operatorname{Arg} (c_1 : c_2) = \operatorname{Arg} c_1 - \operatorname{Arg} c_2.$$

From the last result it follows that the angle between the vectors c_1 and c_2, measured from c_2 to c_1 in the anti-clockwise direction (and definable only to the extent of integral multiples of 2π), is equal to $\operatorname{Arg} \frac{c_1}{c_2}$:

$$\widehat{c_2, c_1} = \operatorname{Arg} \frac{c_1}{c_2}.$$

From the rule of multiplication it follows that

$$c^n = | c |^n (\cos n \operatorname{Arg} c + i \sin n \operatorname{Arg} c),$$

where n is a natural number. Obviously, this formula is valid also when $n = 0$; ($c^0 = 1$). Since $c^{-n} = \frac{1}{c^n}$, we get

$$c^{-n} = | c |^{-n} [\cos (-n \operatorname{Arg} c) + i \sin (-n \operatorname{Arg} c)].$$

Hence, for any integral m the formula

$$c^m = |c|^m (\cos m \operatorname{Arg} c + i \sin m \operatorname{Arg} c)$$

holds. If p and q are integers, $q \geqslant 2$, and the fraction p/q is irreducible, then the right-hand side of the formula

$$c^{p/q} = \sqrt[q]{c^p} = \sqrt[q]{|c|^p} \left[\cos\left(\frac{p}{q} \operatorname{Arg} c\right) + i \sin\left(\frac{p}{q} \operatorname{Arg} c\right)\right],$$

where $\sqrt[q]{|c|^p}$ denotes the positive value of the power $|c|^{p/q}$, yields q different values of the power $c^{p/q}$.

In order to obtain all these values it would be sufficient, after fixing any one value φ of Arg c, to substitute in the right-hand side all the q values of Arg c, namely $\varphi, \varphi + 2\pi, \ldots, \varphi + (q-1)\,2\pi$.

3. THE LIMIT OF A SEQUENCE

The sequence of complex numbers $\{c_n = a_n + ib_n\}$ is said to be *convergent to the limit* $c = a + ib$ (in short, $\lim_{n\to\infty} c_n = c$ or $c_n \to c$ as $n \to \infty$), if, for any $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$ such that $|c_n - c| < \varepsilon$ for $n > N(\varepsilon)$. Since $|a_n - a| \leqslant |c_n - c| < \varepsilon$ and $|b_n - b| \leqslant |c_n - c| < \varepsilon$ for $n > N(\varepsilon)$, hence $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Thus, the last two relations follow from the relation $\lim_{n\to\infty} (a_n + ib_n) = a + bi$. Conversely, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $|a_n - a| < \frac{\varepsilon}{\sqrt{2}}$ and $|b_n - b| < \frac{\varepsilon}{\sqrt{2}}$ for $n > N_1(\varepsilon)$. Hence

$$|a_n + ib_n - (a+ib)| = |c_n - c| = \sqrt{(a_n-a)^2 + (b_n-b)^2} < \varepsilon$$

for $n > N_1(\varepsilon)$, that is, $\lim_{n\to\infty} c_n = c$.

Thus, the relation $\lim_{n\to\infty} (a_n + ib_n) = a + bi$ *is equivalent to the two relations:* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. This fact enables us to apply the entire theory of the limits of sequences of real numbers to sequences of complex numbers. For instance, we obtain the following necessary and sufficient condition for convergence (*the Cauchy criterion*): *For every* $\varepsilon > 0$, *there exists an* $N(\varepsilon)$ *such that* $|c_{n+p} - c_n| < \varepsilon$, if $n > N(\varepsilon)$ *and* p *is any arbitrary natural number.* Further, if $\lim_{n\to\infty} c'_n = c'$ and

$\lim\limits_{n\to\infty} c_n'' = c''$, then

$$\lim_{n\to\infty} (c_n' \pm c_n'') = c' \pm c'', \quad \lim_{n\to\infty} (c_n' \cdot c_n'') = c' \cdot c'', \quad \lim_{n\to\infty} \frac{c_n'}{c_n''} = \frac{c'}{c''}$$

[the last result is valid provided that $c_n'' \neq 0$ $(n = 1, 2, \ldots)$ and $c'' \neq 0$].

We shall call the interior of the circle with centre c and an arbitrary radius ρ as the *ρ-neighbourhood of the point c*. Evidently, the point z belongs to this neighbourhood if, and only if, $|z - c| < \rho$. Now, we can give the following geometrical interpretation to the definition of the limit of the sequence $\{c_n\}$. *The sequence $\{c_n\}$ is said to be converging to the limit c if, for any $\varepsilon > 0$, all points of the sequence, beginning with a certain particular number, belong to the ε-neighbourhood of the point c.*

We leave it to the reader to prove that $\lim\limits_{n\to\infty} |c_n| = |c|$ always follows from $\lim\limits_{n\to\infty} c_n = c$.

If, in addition, $c \neq 0$, then there exists a sequence of values of the arguments of c_n, whose limit is equal to one of the values of the argument of c. For such a sequence we can take the sequence of the principal values of the arguments, excluding the case when $c < 0$ (among c_n one can find infinitely many points lying above as well as below the real axis). The above-mentioned property of the arguments of the sequence $\{c_n\}$ is conventionally written as $\lim\limits_{n\to\infty} \operatorname{Arg} c_n = \operatorname{Arg} c$. Consequently, if the last condition is satisfied and, besides, if $\lim\limits_{n\to\infty} |c_n| = |c|$, then $\lim\limits_{n\to\infty} c_n = c$.

4. THE POINT AT INFINITY AND THE STEREOGRAPHIC PROJECTION

For the requirements of the theory of analytic functions we have to add to the *proper* (finite) complex numbers described above an *improper* (infinite) complex number denoted by the symbol ∞; this number is called the *infinity* or the *point at infinity*. Operations with the point at infinity are based on the following definitions and rules.

The exterior of the circle with its centre at the origin of coordinates and of radius ρ is called the ρ-neighbourhood of the point ∞. Evidently, the point z belongs to this neighbourhood if, and only if, $|z| > \rho$.

The sequence $\{c_n\}$ is said to be *converging to* ∞ (i.e. $\lim\limits_{n\to\infty} c_n = \infty$) if, for any $\rho > 0$, all its points, beginning with a certain number, belong to the ρ-neighbourhood of the point at infinity. In other words, for any $\rho > 0$, there exists an $N(\rho) > 0$ such that $|c_n| > \rho$, if $n > N(\rho)$; hence, the condition $\lim\limits_{n\to\infty} c_n = \infty$ is equivalent to the condition $\lim\limits_{n\to\infty} |c_n| = +\infty$.

We also note that when $c_n \neq 0$, the condition $\lim\limits_{n\to\infty} c_n = \infty$ is equivalent to the condition $\lim\limits_{n\to\infty} \frac{1}{c_n} = 0$.

For the improper complex number the concept of real and imaginary parts as well as the concept of the argument are not introduced ; more precisely, these concepts are, in this case, without any significance (we note here that the concept of the argument has no significance for the number 0 also). For the modulus of the number ∞, we use the symbol $+\infty : |\infty| = +\infty$.

The following operations involving ∞ and the proper complex numbers a and α $(\alpha \neq 0)$ hold by way of definition :

$$\infty \pm a = a \pm \infty = \infty, \quad \infty \cdot \alpha = \alpha \cdot \infty = \infty \cdot \infty = \infty,$$

$$\frac{a}{\infty} = 0, \quad \frac{\infty}{a} = \infty, \quad \frac{\alpha}{0} = \infty.$$

Operations $\infty \pm \infty$, $0 \cdot \infty$, $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are indeterminate.

In order to obtain a geometrical representation of the number ∞, we resort to the representation of the complex numbers by the points of a sphere.

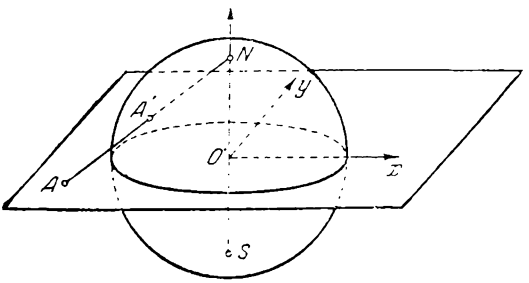


Fig. 3.

For this purpose we take a sphere of unit radius, with its centre at O, in the complex plane z (Fig. 3). For the sake of clarity we take recourse to the geographical terminology. The circle in which the sphere interesects the complex plane will be called the equator, the straight line passing through O and perpendicular to this plane will be called the axis of the sphere, and the points N and S at which the axis meets the sphere—the north and south poles, respectively. We shall also use the concepts and terms like northern and southern hemispheres, meridians, parallels, latitude φ and longitude λ. The latitude is measured from the equator between the limits $-\frac{\pi}{2}$ (south pole) and $\frac{\pi}{2}$ (north pole). The longitude is measured along the equator, starting with the positive part of

the x-axis, between the limits $-\pi$ (exclusive) and π (inclusive), positive direction here being the anti-clockwise direction (looking from the north pole to the equator). We shall now join the point N with different points of the sphere, with rectilinear rays originating at N and mark on each ray its point of incidence on the plane. Thus, all points of the sphere, excepting the point N, are projected on the plane. This projection (central projection, with centre at N) is called the *stereographic projection*. Such a projection was first used in astronomy in ancient times and later in geography for the purpose of mapping the celestial or the terrestrial sphere on a plane. With the help of stereographic projection each point of the sphere (excepting N) can be represented as a corresponding point in the plane and also as the image of the complex number represented by this point of the plane. We shall now explain the nature of the connection between the latitude and the longitude of a point of the sphere corresponding to the complex number $z = r(\cos\alpha + i\sin\alpha)$ $(\alpha = \arg z)$ and its modulus and the argument.

We find from Fig. 4 that $\widehat{ONA'} = \frac{\pi}{4} + \frac{\varphi}{2}$ and, hence, $r = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$. It is evident that $\alpha = \lambda$; hence, $\varphi = 2\arctan r - \frac{\pi}{2}$, $\lambda = \alpha$. If the condition $\lim\limits_{n\to\infty} z_n = \infty$ holds for the sequence $\{z_n\}$ in which $|z_n| = r_n$, then we have $\lim\limits_{n\to\infty} r_n = +\infty$; hence, $\lim\limits_{n\to\infty} \varphi_n = \lim\limits_{n\to\infty}\left(2\arctan r_n - \frac{\pi}{2}\right) = \frac{\pi}{2}$. Thus, the points of the sphere, corresponding to the numbers z_n, indefinitely converge to the north pole N. The converse is also true. If $\varphi_n \to \frac{\pi}{2}$ (whatever be the value of the longitude λ_n), then $r_n = \tan\left(\frac{\pi}{4} + \frac{\varphi_n}{2}\right) \to +\infty$; consequently, $\lim\limits_{n\to\infty} z_n = \infty$. Hence, naturally, we consider the point N as the representation of the point at infinity on the sphere. In complete agreement with this condition is the fact that the neighbourhood $|z| > \rho$, on the plane, of the point at infinity is mapped on to the sphere by the domain $\varphi > 2\arctan\rho - \frac{\pi}{2}$ about the pole; for $\rho \to \infty$ this domain contracts into the north pole itself.

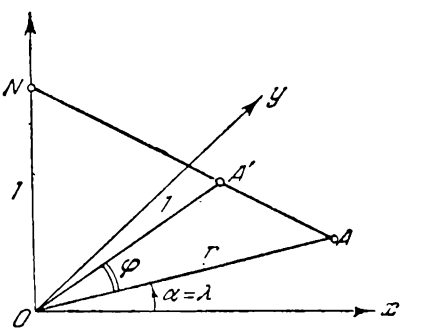


Fig. 4.

The complex plane to which the point at infinity has been uniquely adjoined is called the *extended complex plane* or simply the *extended plane*. A graphic representation of the extended plane is the entire sphere. The

complex plane formed only by the proper (finite) points is called the *finite complex plane*, or simply the *finite plane*. From what has been said above it follows that the finite plane can be visually represented by a sphere, with one point, viz. the point N, eliminated.

We shall now consider the mirror image of the sphere in its equatorial plane. Here, the sphere passes over into itself such that the northern hemisphere passes over into the southern (and conversely), the north pole into the south (and conversely), while the equator remains its own self. In general, a point with geographical coordinates (φ, λ) passes over into the point $(-\varphi, \lambda)$.

This image of the sphere onto itself would correspond to the image of the extended plane onto itself and, here, the point z with coordinates $r = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$ and $\alpha = \lambda$ passes over to the point z' with coordinates $r' = \tan\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)$ and $\alpha' = \lambda = \alpha$. Evidently, z and z' are connected by the relation $\bar{z} \cdot z' = 1$, that is, $z' = \frac{1}{\bar{z}}$. This transformation maps the exterior of the unit circle onto its interior (and conversely) and, in particular, the point ∞ onto 0 (and conversely). The unit circle is here mapped onto itself. The transformation $z' = \frac{1}{\bar{z}}$ may, therefore, be regarded as a transformation of symmetry of the extended plane relative to the unit circle, or a mirror image in the unit circle. This point of view appears justified on examining what happens in this case on the sphere. Later on (in Chapter 3, Art. 8) we shall give a more general treatment of the transformation of symmetry of the extended plane with respect to any arbitrary circle of the plane.

5. POINT SETS IN A PLANE

We shall recall here certain definitions and properties of the point sets in a plane, well known in mathematical analysis; to some extent, we shall also supplement these for use in the sequel.

The point z_0 is called the *limit point* of a certain set E if every neighbourhood of this point contains an infinite number of points belonging to E.

The point set E in the plane is said to be *bounded* if all its points are contained within a certain circle with its centre at the origin of coordinates. The set F (bounded or unbounded) is said to be *closed* if not a single point, not belonging to F, can be its limit point. In other words, a closed set either does not have limit points at all or else contains all of them within itself.

Every infinite bounded point set E has at least one limit point (the Bolzano-Weierstrass theorem). If the infinite set E is unbounded, then there are only two possibilities : either a certain circle $|z| \leqslant R$ would contain the infinite set of points E and consequently this set would also have a limit point or in every such circle there would be found only a finite number of points of E ; then their infinite set will lie in any neighbourhood $|z| > R$ of the point at infinity and hence ∞ will be the limit point of E. Thus, in the extended plane every infinite set has at least one limit point (which is either finite or infinite).

Let $\{K\}$ *be a set of circles forming a covering of a bounded closed set* F, that is, of such circles that for every point $z \in F$ there exists at least one circle of the set $\{K\}$ containing this point in its interior ; then, from the set $\{K\}$ we can isolate a finite number of circles $K_1, K_2, \ldots, K_n$ also forming the covering of F *(Heine-Borel theorem).*

Let E be any set of points in a plane. The lower bound of the distances of a point ζ from all possible points of E: $\rho(\zeta, E) = \min |\zeta - z|$, $z \in E$, is called the *distance of the point* ζ *from this set.* If $\rho(\zeta, E) = 0$, then either $\zeta \in E$, or $\zeta \in E$, but then E contains points as close as desired to ζ, that is, ζ is the limit point of E. When E is a closed set, a point not belonging to E cannot be a limit point of E. *Hence, from the condition* $\zeta \bar{\in} E$, *it follows that* $\rho(\zeta, E) > 0$.

Let E and F be *two point sets.* The distance between them is called the lower bound of the distances between all possible pairs of points z' and z'', such that $z' \in E$ and $z'' \in F$; $\rho(E, F) = \min |z' - z''|$. The distance between E and F may be equal to zero even when E and F have no common points. But if both the sets are closed and at least one of them, say E, is bounded, *then because of the fact that E and F do not have common points it follows that* $\rho(E, F) > 0$.

In fact, if $z' \in E$, then $z' \bar{\in} F$ and, consequently $\rho(z', F) > 0$. We draw a circle of radius $\rho(z', F)$ with z' as its centre. None of the points of F will lie in the interior of this circle. The set of circles with the same centre but radii equal to one half of the previous one forms a covering of the set E. According to Heine-Borel theorem, there exists a finite number of circles $K_1, K_2, \ldots, K_n$ with centres $z_1', z_2', \ldots, z_n'$ and radii $\frac{1}{2}\rho(z_1', F), \frac{1}{2}\rho(z_2', F), \ldots, \frac{1}{2}\rho(z_n', F)$, covering E. We denote the smallest of these radii by δ ($\delta > 0$). Let $z' \in E$, then $z' \in K_j$ and since a concentric circle with double the radius $2 \cdot \frac{1}{2}\rho(z_j', F)$ does not contain any point of F whatsoever, we have the relation $|z' - z''| \geqslant \frac{1}{2}\rho(z_j', F) \geqslant \delta$, for any point $z'' \in F$. Hence, we also have $\rho(z', z'') = \min |z' - z''| \geqslant \delta > 0$ and

this is what was to be proved.

A point of a certain set E is called an *interior point* (in relation to this set) if there exists a neighbourhood of the point, contained in E. The set E consisting only of interior points is called an *open set*. Points that are limit points for the open set but do not belong to this set are called *boundary points*, and the aggregate of these forms the *boundary* Γ of the set E. The boundary itself is a closed set. The set $\overline{E} = E \cup \Gamma$ obtained by the unification of the set E and its boundary Γ and called the *closure* of E is a closed set, and so is the set of all points of the plane not belonging to E. The latter can be decomposed into two subsets : the boundary Γ of the set E and the set E_1 of points, neither belonging to E nor being its limit points—the so-called *exterior* points. For every exterior point there exists a neighbourhood not belonging to E. Such a neighbourhood is completed only by exterior points. Hence the set E_1 of all exterior points of the set E is itself an open set. The most important instance of an open set is a *domain*. An open set E is called a *domain* if any two points of E may be joined by a broken line contained in E (in a particular case, this broken line may reduce to a rectilinear segment). Domains are most often denoted by the letters G (initial letter of the German word *Gebiet*), D (initial letter of the French word *Domaine*) and B (initial letter of the German word *Bereich*).

Examples : *a*) All points z satisfying the inequality $|z-z_0| < \rho$ (ρ being a fixed positive number) form a domain G—the interior of a circle with its centre at z_0 and radius ρ. The boundary of this domain is the circumference $\Gamma : |z - z_0| = \rho$. Exterior points are characterized by the inequality $|z-z_0| > \rho$; the totality of these points also form a domain G_1, the exterior of the circle. The point ∞ is also an exterior point relative to G and hence belongs to G_1. The boundary of the domains G_1 is the same periphery Γ. We further add that every point $z \in G$ is an external point in relation to G_1 such that the totality of all points exterior to G_1 coincides with G.

b) Let Γ, given by the equation $Ax + By + C = 0$ (A, B and C are real numbers and $A^2 + B^2 \neq 0$), be any straight line in the plane z. All points satisfying the inequality $Ax + By + C > 0$ and all points satisfying the inequality $Ax + By + C < 0$ form one or the other of the two distinct domains G_1 and G_2, having a common boundary Γ. These domains, G_1 and G_2, are called *half-planes* (bounded by the straight line Γ). Each of these consists of points exterior to the other. The point $z = \infty$ is the boundary point for both G_1 and G_2 (Γ passes through this point).

c) The set of points $R_1 < |z-z_0| < R_2$ is a domain (a circular ring),

with its boundary consisting of two concentric circles; $\Gamma_1 : | z - z_0 | = R_1$ and $\Gamma_2 : | z - z_0 | = R_2$. The totality of the exterior points can be here decomposed into two domains—the interior of the circle : $| z - z_0 | < R_1$ and the exterior of the circle : $| z - z_0 | > R_2$.

2

FUNCTIONS OF A COMPLEX VARIABLE. DERIVATIVE AND ITS GEOMETRICAL AND HYDROMECHANICAL INTERPRETATION

1. FUNCTIONS OF A COMPLEX VARIABLE

The concept of a function of a complex variable is a particular case of the general mathematical concept of a function. Thus, if E is a certain point set in the complex plane (z) and if to every z there corresponds one or more complex numbers w, it is said that *a function of the complex variable* z with its values equal to w is defined in E; in brief, $w = f(z)$. If to every z there corresponds only one value of w, the function is called *one-valued* and if to every z there correspond more than one values of w, it is called *many-valued*. Thus, for instance, $w = z^n$ (n, a natural number), $w = |z|$, $w = \bar{z}$, $w = \operatorname{Re} z$ and $w = \operatorname{Im} z$ are one-valued functions defined on the whole (finite) plane; $w = \sqrt[n]{z}$ is a many-valued function, actually n-valued, also defined on the whole plane; $w = \operatorname{Arg} z$ is a many-valued function (infinite-valued) defined on the set of all points different from zero. If E is confined to the real axis, $z = x$ is a real variable; further, if all the values of w are also real, we have the concept of a real function of one real variable as a special case of the functions of a complex variable.

In the general case, we put $z = x + iy$ and $w = u + iv$. Then the statement that '*the function* $w = f(z)$ *(say, one-valued) is defined in* E' is equilvalent to the statement that '*to each point of* E *with coordinates* x *and* y *there correspond the real numbers* u *and* v'. In other words, two real functions $u = \varphi(x, y)$ and $v = \psi(x, y)$ of two real variables x and y are defined in E. Thus, one complex relation $w = f(z)$ is equivalent to two relations: $u = \varphi(x, y)$ and $v = \psi(x, y)$. For example, $w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$ is equivalent to $u = x^2 - y^2$ and $v = 2xy$.

2. LIMIT OF A FUNCTION AT A POINT

Let $w = f(z)$ be a one-valued function defined in the set E, z_0 being a limit point of this set. If, for a fixed complex number A and for

every $\varepsilon > 0$, we can find $\delta(\varepsilon) > 0$ such that $|f(z) - A| < \varepsilon$ for $|z - z_0| < \delta(\varepsilon)$, $z \in E$ (and $z \neq z_0$), then we say that $f(z)$ *approaches the limit A as z tends to* z_0, and we write

$$\lim_{z \to z_0,\ z \in E} f(z) = A.$$

In what follows the specific indication $z \in E$ will always be left out when such an omission is not likely to cause any confusion.

Putting $A = B + iC$, $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and arguing in the same manner as in Art. 3 of Chapter 1, we find that the preceding complex relation is equivalent to two real relations

$$\lim_{x \to x_0,\ y \to y_0} u(x, y) = B \quad \text{and} \quad \lim_{x \to x_0,\ y \to y_0} v(x, y) = C.$$

This remark shows that the simplest propositions relating to limits of functions of real variables can, without alteration, be extended to limits of functions of a complex variable. For example, if the functions $g(z)$ and $h(z)$ are defined in one and the same set E and for them the limits

$$\lim_{z \to z_0} g(z) = A_1 \quad \text{and} \quad \lim_{z \to z_0} h(z) = A_2$$

exist, then

$$\lim_{z \to z_0} [g(z) \pm h(z)] = A_1 \pm A_2, \quad \lim_{z \to z_0} [g(z) \quad h(z)] = A_1 \cdot A_2 \text{ and}$$

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{A_1}{A_2}$$

(the last one being true for $A_2 \neq 0$).

In an analogous manner we can study the case where one chooses the point at infinity instead of a finite point. Indeed, if $z = \infty$ is a limit point of E and for any fixed complex number A and for any $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that $|f(z) - A| < \varepsilon$ for $|z| > N(\varepsilon)$ ($z \in E$), then we say that $f(z)$ *approaches the limit A as z tends to* ∞ ; we then write

$$\lim_{z \to \infty} f(z) = A.$$

It is obvious that the difference between this and the previous case lies only in that here instead of the neighbourhood $|z - z_0| < \delta(\varepsilon)$ of the finite point z_0 we take the neighbourhood $|z| > N(\varepsilon)$ of the point at infinity.

Finally, if z_0 is any limit point of the set E (finite or at infinity) and for any number $N > 0$ we can find a neighbourhood of the point such that the inequality $|f(z)| > N$ is satisfied; further, if z belongs to this neighbourhood ($z \in E$ and $z \neq z_0$), then we say that $f(z)$ *tends to* ∞ *as* z *tends to* z_0; we then write

$$\lim_{z \to z_0} f(z) = \infty.$$

All these three special cases of the limit of a function can be covered by a single general definition: *let z_0 be a limit point of the set E (finite or at infinity) and A a complex number (proper or improper); then, if to every neighbourhood U of the point A there corresponds a neighbourhood u_0 of the point z_0 such that $f(z)$ belongs to U if z belongs to u_0 (and, moreover, $z \in E$ and $z \neq z_0$), then we say that $f(z)$ approaches the limit A as z tends to z_0 and we write*

$$\lim_{z \to z_0,\ z \in E} f(z) = A.$$

In the case of such a general definition of the limit, when $A = \infty$ we cannot, without reservations, use the theorems on the limits of a sum, difference, product and quotient of the functions, because the operations $\infty \pm \infty$, $a \cdot \infty$, $\frac{\infty}{\infty}$ are indeterminate.

3. CONTINUITY

If the limit point z_0 (finite or at infinity) of the set E belongs to the set and if the condition

$$\lim_{z \to z_0,\ z \in E} f(z) = f(z_0) \quad [f(z_0) \neq \infty]$$

holds for the function $f(z) = u(x, y) + iv(x, y)$ defined in E, then $f(z)$ is said to be *continuous at the point* z_0 (with respect to the set E).

If $f(z)$ is continuous at each point of the set E, it is said to be *continuous in the set E.*

According to Art. 2, the condition of continuity of $f(z)$ at the point $z_0 = x_0 + iy_0$ is equivalent to the two relations

$$\lim_{x \to x_0,\ y \to y_0} u(x, y) = u(x_0, y_0) \quad \text{and} \quad \lim_{x \to x_0,\ y \to y_0} v(x, y) = v(x_0, y_0),$$

expressing the continuity of the two real functions $u(x, y)$ and $v(x, y)$ at the same point.

Thus, *a function of a complex variable is continuous at the point z_0 if, and only if, its real and imaginary parts, considered as functions of the real variables x and y, are continuous at that point.*

Hence it follows that several properties of continuous functions of two real variables can be directly transferred to continuous functions of a complex variable.

Indeed, the sum, the difference, the product and the quotient of two continuous functions are themselves continuous (in the case of quotient, those points where the denominator vanishes are excluded). Further, if the function $w = f(z)$ is continuous in the set E and if its values belong to the set F in which the function $\zeta = \varphi(w)$ is continuous, then the composite function $\zeta = \varphi[f(z)] = F(z)$ is continuous in E.

Let the set E be bounded and closed. Then, every function $w = f(z)$, continuous in E, is bounded in this set, that is, it satisfies a relation of the type $|f(z)| \leqslant C < \infty$, $z \in E$; its modulus attains its upper and lower bounds in E and, finally, $f(z)$ is *uniformly continuous in E.* The last statement means that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ for any two points of the set for which $|z_1 - z_2| < \delta(\varepsilon)$. All these properties follow from the relevant theorems pertaining to functions of two real variables, continuous in bounded closed sets. It is, however, not difficult to prove them directly, repeating almost without any change the proofs given in analysis.

In defining continuity, it was assumed that $f(z_0) \neq \infty$. While studying mappings with analytic functions it is expedient to remove this restriction and to consider the function as continuous at the point z_0, where $f(z_0) = \infty$, if

$$\lim_{z \to z_0,\ z \in E} f(z) = \infty.$$

In this case we call it a function with *generalized continuity.* The various properties mentioned above cannot be extended to functions with generalized continuity.

Example. *The function $f(z) = \frac{1}{z}$ for $z \neq 0$ and $z \neq \infty$, reducing to zero at the point $z = \infty$ and to infinity at the point $z = 0$, is a function with generalized continuity in the extended plane. Indeed, for this function*

$$\lim_{z \to \infty} f(z) = 0 = f(\infty) \quad \text{and} \quad \lim_{z \to 0} f(z) = \infty = f(0).$$

4. CONTINUOUS CURVE

The concept of a continuous curve is a generalization of the graphic representation of a curve as the trajectory of a moving point. When the

function $z = \lambda(t)$ of a real variable (parameter) t is continuous on a certain segment $[\alpha, \beta]$, we say that this function defines a *continuous curve* (also a line or an arc). The values of the function are called the *points of the curve* and the equation $z = \lambda(t)$ the *equation of the curve* (in parametric form). For each curve we can fix one of the two mutually opposite directions of the curve corresponding to an increase or a decrease of the parameter. In the first case, $\lambda(\alpha)$ is the initial point of the curve and $\lambda(\beta)$ the terminal point, while in the second case these points interchange their roles. A curve in which the initial and the terminal points coincide is called a *closed* curve.

If one and the same point z of the curve corresponds to two or more different values of the parameter, of which at least one is distinct from α and β, we call such a point as a *multiple point*. A curve not having any multiple points is called a *simple curve* or a *Jordan curve*.

Two continuous curves $z = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$ and $z = \mu(\tau)$ $(\gamma \leqslant \tau \leqslant \delta)$ are regarded as identical if, and only if, there exists a function $\tau = \varphi(t)$, continuous and monotonic in the segment $[\alpha, \beta]$, such that $\varphi(\alpha) = \gamma$, $\varphi(\beta) = \delta$ (or $\varphi(\alpha) = \delta$, $\varphi(\beta) = \gamma$) and $z = \mu[\varphi(t)] = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$.

It is not difficult to show that the set of all the points of a continuous curve is a closed set (we leave its proof to the reader).

Examples of curves. *a*) The equation $z = t$ $(-1 \leqslant t \leqslant 1)$ defines a curve represented by the segment $-1 \leqslant x \leqslant 1$ of the real axis. For its direction, corresponding to the increase of the parameter, the initial point will be -1 and the terminal point $+1$; the curve is not closed. It does not possess multiple points; hence, it is a Jordan curve (arc).

b) $z = \cos t$ $(0 \leqslant t \leqslant \pi)$ is a curve identical to the above; here, the direction corresponds to a decreasing parameter.

c) $z = \cos t$ $(0 \leqslant t \leqslant 2\pi)$ is a curve represented by the same segment of the real axis $-1 \leqslant x \leqslant 1$, but it is not identical to the previous one. This is, indeed, a closed curve because $t = 0$ and $t = 2\pi$ correspond to one and the same point $z = 1$. Moreover, one and the same point $z = \cos t$ here corresponds to different values of the parameter t and $2\pi - t$ $(0 < t < 2\pi)$; hence, the curve does have multiple points and is not a Jordan curve.

The difference between the continuous curves *a*), *b*) and *c*) lies in that for the first two the point z runs once through the segment $[-1, 1]$ as the parameter runs through the entire interval of its variation, while in the case of the last curve the point z under the same conditions runs twice through the segment.

d) Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be rectilinear segments oriented in a definite manner and so placed on the plane that the end of each segment

Δ_j $(j=1, 2, \ldots, n-1)$ coincides with the beginning of the following segment Δ_{j+1}. Denoting by a_j the complex number represented by the vector Δ_j and by z_0 the initial point of the segment Δ_1, we can obtain in the segment $0 \leqslant t \leqslant n$ a very simple continuous function defining the curve that represents the totality of the given segments :

$$z = z_0 + a_1 + \ldots + a_{j-1} + a_j (t-j+1) \quad (j-1 \leqslant t \leqslant j, \quad j=1, 2, \ldots, n).$$

This curve is called a *broken* curve and the segments Δ_j its *sections*. The broken curve is closed or not closed, depending on whether the end of the segment Δ_n coincides or does not concide with the initial point of Δ_1. It will be a Jordan curve only when there are no points of self-intersection, that is, provided that two adjacent parts Δ_j and Δ_{j+1} can have one and only one common point (in the case of a closed broken curve, the parts Δ_n and $\Delta_1 = \Delta_{n+1}$ are adjacent).

We shall now show that if any two points z_0 and z' of an open set E can be joined by a continuous curve L, which itself is contained in E, then they can also be joined by a broken line, again contained in E; hence it follows that E is a domain (*cf.* Art. 5 of Chapter 1).

Now, let $z = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$ be the equation of L, with $z_0 = \lambda(\alpha)$ and $z' = \lambda(\beta)$. We denote by δ $(\delta > 0)$ the distance between L and the boundary Γ of the set E. Utilizing the uniform continuity of $\lambda(t)$ on the segment $[\alpha, \beta]$, we divide this segment by the points $t_0 = \alpha < t_1 < t_2 < \ldots < t_n = \beta$ into such small segments that the conditions $|\lambda(t_{j+1}) - \lambda(t_j)| < \delta$ $(j=0, 1, \ldots, n-1)$ are satisfied; we then join every pair of adjacent points of the curves $z_j = \lambda(t_j)$ and $z_{j+1} = \lambda(t_{j+1})$ by the chords Δ_j. All these chords $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$ are evidently contained in E. They constitute together a broken curve Δ inscribed in L, joining z_0 with z'. Hence, E is a domain.

We now give two theorems on Jordan curves and mappings, and accept these without proof (proofs are available in books on topology).

Jordan's theorem. *Every closed Jordan curve* Γ *divides the whole plane into two distinct domains* G_1 *and* G_2, *itself being the boundary common to them. Here, one of the domains is bounded* (*called the interior of* Γ) *while the other is not bounded* (*called the exterior of* Γ).

We shall denote the interior by the notation $I(\Gamma)$ and the exterior by $E(\Gamma)$ (I and E are the initial letters of the French words *intérieur* meaning interior and *extérieur* meaning exterior).

The simplest illustration of the Jordan theorem is afforded by the interior ($|z-z_0|<\rho$) and the exterior ($|z-z_0|>\rho$) of the circle Γ: $|z-z_0|=\rho$ (in parametric form, $z=z_0+\rho(\cos t+i\sin t)$ $(0\leqslant t\leqslant 2\pi)$).

Let G be any domain. If for any closed Jordan curve γ, pertaining to G, the interior of γ also belongs to G, the domain G is called *simply-connected* (in the finite plane). An example of a simply-connected domain is the interior of a circle. The exterior of a circle or a circular ring is not simply-connected in the finite plane, because for each of these domains we can find a circle belonging to the domain such that its interior does not entirely belong to the domain. To meet the needs of the theory of conformal mapping, the concept of a simply-connected domain has to be generalized. In fact, the domain G of the extended plane is called *simply-connected* (in the extended plane) if, for any closed Jordan curve γ, belonging to G, the interior or the exterior of γ also belongs to G. All other domains are called *multiply-connected.* Of course, the domain simply-connected in respect of the finite plane is also simply-connected in respect of the extended plane. The converse is, in general, not true. Thus, the exterior of the circle to which also belongs the point at infinity in the extended plane is simply-connected in respect of the extended plane, although it is not simply-connected in respect of the finite plane. But a circular ring is not simply-connected both in respect of the finite and the extended plane, that is, it is a multiply-connected domain.

Theorem on one-to-one and continuous mappings. *Let G be the domain of the extended plane and let $w=f(z)$ be a function, with generalized continuity in G, which maps G one-to-one onto some set D.* Then, *D is also a domain and the function $z=f^{-1}(w)$, inverse to $f(z)$, is a function with a generalized continuity in the domain D. If $f(z)$ is defined also on the boundary Γ of the domain G under these very assumptions such that it is a function with generalized continuity in the closed domain $\bar{G}$, it maps Γ onto the boundary Δ of the domain D; in other words, the boundary of the image of the domain G coincides with the image of the boundary of the same domain.*

Later on we shall examine this theorem under the additional assumption that $f(z)$ is analytic in the domain G (*cf.* Arts. 1, 4 and 7 of Chapter 10).

Now, at the end of this article we generalize the concept of a continuous curve. Let $z=\lambda(t)$ be a function with generalized continuity on the segment $[\alpha, \beta]$ (which may be here infinite on one or both the sides), such that we can say it defines a *generalized continuous curve* in the extended plane. If $z=\lambda(t)$ does not become ∞ at any point whatsoever of the segment $[\alpha, \beta]$, the generalized curve does not pass through the point at

infinity. The concepts of the initial and the terminal points of the curve, of a closed curve, of multiple points and of Jordan curve can naturally be extended to the case of a generalized continuous curve.

Examples of generalized Jordan curves are: the straight line $z = (\alpha t+\beta) + i(\gamma t+\delta)$ $(-\infty \leqslant t \leqslant +\infty)$, where $\alpha^2 + \gamma^2 \neq 0$ and the function z becomes ∞ for $t = \pm\infty$; the parabola $z = (\alpha t^2+\beta t+\gamma) + i(\delta t+\varepsilon)$ where $\alpha \neq 0$ and z becomes ∞ again for $t = \pm\infty$. The hyperbola $z = \frac{a(1+t^2)+2ibt}{1-t^2}$ $(-\infty \leqslant t \leqslant +\infty)$, where $a \neq 0$ and $b \neq 0$, is a generalized continuous curve, but not a Jordan curve because the point $z = \infty$ is a multiple point of the curve (this point corresponds to two different values of t, viz. ± 1).

5. DERIVATIVE AND DIFFERENTIAL

Let $f(z)$ be a one-valued function of a complex variable, defined in some set E and let z_0 be a limit point of the set.

We now form the difference ratio $\frac{f(z)-f(z_0)}{z-z_0}$. Evidently this ratio represents a function of z, defined for all points of the set E other than z_0.

If there exists the limit

$$\lim_{z \to z_0,\ z \in E} \frac{f(z)-f(z_0)}{z-z_0},$$

it is called *the derivative of the function $f(z)$, relative to the set E, at the point z_0* and is denoted by the symbol $f'_E(z_0)$ or simply by $f'(z_0)$. The function $f(z)$ itself possessing a derivative is said to be *differentiable, or monogenic, relative to the set E, at the point z_0*.

In the particular case when E is an interval of the real axis (finite or infinite), $f(z)$ is a function of the real variable $z = x$, taking up, generally speaking, complex values $f(z) = f(x) = \varphi(x) + i\psi(x)$. If $\psi(x) \equiv 0$, that is, if the values of $f(z)$ are real, this definition of a derivative and of differentiability evidently coincides with the usual definitions in differential calculus. If $\psi(x) \not\equiv 0$, then writing $\frac{f(x)-f(x_0)}{x-x_0}$ in the form $\frac{\varphi(x)-\varphi(x_0)}{x-x_0} + i\frac{\psi(x)-\psi(x_0)}{x-x_0}$, we conclude, in accordance with Art. 2, that the derivative $f'(x)$ exists if, and only if, there exist the derivatives $\varphi'(x)$ and $\psi'(x)$, where $f'(x) = \varphi'(x) + i\psi'(x)$. Thus, for instance, if $f(x) = a\cos x + ib\sin x$, then $f'(x) = -a\sin x + ib\cos x$.

Denoting $f(z) - f(z_0)$ by $\Delta f(z)$ (increment of the function) and $z - z_0$ by Δz (increment of the independent variable), we write the

condition of differentiability as

$$\frac{\Delta f(z)}{\Delta z} = f'(z_0) + \varepsilon(z_0, \Delta z),$$

where $\varepsilon(z_0, \Delta z) \to 0$ as $\Delta z \to 0$ $(z \in E)$. Hence it follows that the increment of a differentiable function may be written in the form

$$\Delta f(z) = A \cdot \Delta z + \varepsilon(z_0, \Delta z) \cdot \Delta z \qquad [A = f'(z_0)] \qquad (1)$$

with A, independent of Δz, and ε tending to zero along with Δz. Conversely, any function, whose increment can be represented in the form (1), with the same conditions relating to A and $\varepsilon(z_0, \Delta z)$, is differentiable and its derivative is equal to A. Indeed, it follows from (1) that the limit

$$\lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z} \qquad (\Delta z = z - z_0;\ z \in E)$$

exists and is equal to A. Thus, the representability of the increment of a function in the form (1), with A not depending on Δz and ε tending to zero along with Δz, is the necessary and sufficient condition for the differentiability of the function. It follows directly from (1) that the function differentiable at the point $z_0 \in E$ is continuous at this point (in relation to this set).

Replacing Δz by dz (the differential of the independent variable) and $A \cdot \Delta z = f'_E(z_0)\, dz$ by $df(z) = d_E f(z)$ (differential of the function), we obtain for the derivative the following expression in terms of the differentials :

$$f'_E(z_0) = \frac{d_E f(z)}{dz}.$$

We shall now illustrate by an example the role played by the set E, relative to which the derivative, in the definition of the concept of differentiability is taken.

Firstly, let E be confined to the real axis and $f(z) = f(x) = x$. Then, the derivative $f'_E(x)$ exists for any $x \in E$ and is equal to unity, that is, the function is differentiable everywhere in E. We now continue the function $f(x)$ on the whole of the complex plane E_1, assuming as before $f(z) = x$. This function is evidently continuous for any value of z and coincides with the original function, when $z \in E$ (that is, when $y = 0$). The difference ratio here is

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(x - x_0)}{(x - x_0) + i(y - y_0)}.$$

This ratio has no limit as $z \to z_0$ (z_0 being any point of the plane), because for $x = x_0$ and $y \neq y_0$ it is equal to zero and for $x \neq x_0$ and $y = y_0$ it is equal to unity. Thus, relative to the plane, the function $f(z)$ is not differentiable at any point.

6. RULES OF DIFFERENTIATION

From the definition of the derivative and the properties of the limits of the functions of a complex variable, it follows that the basic rules of differential calculus can also be extended to derivatives, relative to a set, of the functions of complex variables. These rules are as follows :

1. If $f(z) \equiv c$, then $\dfrac{df(z)}{dz} = 0$.

2. $\dfrac{d[cf(z)]}{dz} = c\dfrac{df(z)}{dz}$.

3. $\dfrac{dz}{dz} = 1$.

4. $\dfrac{d}{dz}[f_1(z)+f_2(z)+\ldots+f_n(z)] = \dfrac{df_1(z)}{dz}+\dfrac{df_2(z)}{dz}+\ldots+\dfrac{df_n(z)}{dz}$.

5. $\dfrac{d}{dz}[f_1(z)\cdot f_2(z)\ldots f_n(z)] = f_2(z)f_3(z)\ldots f_n(z)\dfrac{df_1(z)}{dz} +$

$+ f_1(z)f_3(z)\ldots f_n(z)\dfrac{df_2(z)}{dz}+\ldots+f_1(z)f_2(z)\ldots f_{n-1}(z)\dfrac{df_n(z)}{dz}$.

6. $\dfrac{d}{dz}[f(z)]^n = n[f(z)]^{n-1}\cdot f'(z)$. 6'. $\dfrac{d}{dz}(z^n) = nz^{n-1}$.

7. $\dfrac{d}{dz}(a_0+a_1z+\ldots+a_nz^n) = a_1 + 2a_2z + \ldots + na_nz^{n-1}$.

8. $\dfrac{d}{dz}\left[\dfrac{f_1(z)}{f_2(z)}\right] = \dfrac{f_2(z)\dfrac{df_1(z)}{dz} - f_1(z)\dfrac{df_2(z)}{dz}}{[f_2(z)]^2}$.

Here, all the functions $f(z)$, $f_1(z)$, $f_2(z)$, ... are assumed to be differentiable at any given point z of the set E. The Rule 8 further requires that $f_2(z)$ be nonzero.

9. *Rule for differentiation of composite functions.* We assume that the function $w = f(z)$ is differentiable at the point $z_0 \in E$ and examine the function $Z = \varphi(w)$, defined in the set F of the values of this function and differentiable, relative to this set, at the point $w_0 = f(z_0)$. Then, the composite function $Z = \varphi[f(z)]$ is differentiable, relative to the set E, at the point z_0, and

$$\frac{d_E\varphi[f(z)]}{dz} = \frac{d_F\varphi(w)}{dw}\cdot\frac{d_Ef(z)}{dz}.$$

10. *Rule for differentiation of inverse functions.* Let the function $w = f(z)$ establish a one-to-one correspondence between the points of the two sets E and F, the inverse function $z = \varphi(w)$ being continuous in F. Then, if $f(z)$ is differentiable at the point $z_0 \in E$ and $f'_E(z_0) \neq 0$, then the inverse function $z = \varphi(w)$ is also differentiable at the point $w_0 = f(z_0) \in F$, and

$$\varphi'_F(w_0) = \frac{1}{f'_E(z_0)}.$$

Indeed, by virtue of the one-to-one correspondence in the mapping $w = f(z)$, $z \neq z_0$ when $w \neq w_0$; hence, the difference ratio for the function $\varphi(w)$ may be written in the form

$$\frac{\varphi(w)-\varphi(w_0)}{w-w_0} = \frac{z-z_0}{w-w_0} = \frac{1}{\dfrac{w-w_0}{z-z_0}},$$

and since $z = \varphi(w) \to z_0 = \varphi(w_0)$ as $w \to w_0$, we have

$$\lim_{w\to w_0} \frac{\varphi(w)-\varphi(w_0)}{w-w_0} = \frac{1}{\lim\limits_{z\to z_0} \dfrac{w-w_0}{z-z_0}} = \frac{1}{\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{f'_E(z_0)},$$

as was to be proved.

7. NECESSARY AND SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY AT AN INTERIOR POINT OF A DOMAIN

We shall consider here mainly the functions defined in a certain domain $E = G$ and, instead of $f'_E(z)$ or $\dfrac{d_E f(z)}{dz}$, we shall briefly write $f'(z)$ or $\dfrac{df(z)}{dz}$.

Let $f(z) = u(x, y) + iv(x, y)$. We recall that a function of two real variables, $u(x, y)$, is differentiable at the point (x_0, y_0) of the domain of its definition, if there holds the relation

$$u(x, y) - u(x_0, y_0) = A(x_0, y_0)(x-x_0) + B(x_0, y_0)(y-y_0) + \\ + \varepsilon_1(x, y; x_0, y_0)(x-x_0) + \varepsilon_2(x, y; x_0, y_0)(y-y_0),$$

where

$$\lim_{\substack{x\to x_0,\\ y\to y_0}} \varepsilon_1(x, y; x_0, y_0) = \lim_{\substack{x\to x_0,\\ y\to y_0}} \varepsilon_2(x, y; x_0, y_0) = 0.$$

The coefficients $A(x_0, y_0)$ and $B(x_0, y_0)$ on the right-hand side of the foregoing equation are the partial derivatives of the function $u(x, y)$:

$$A(x_0, y_0) = \left.\frac{\partial u(x, y)}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}, \qquad B(x_0, y_0) = \left.\frac{\partial u(x, y)}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}}.$$

We shall now prove the following important proposition.

Theorem. *In order that the function* $f(z) = u(x, y) + iv(x, y)$, *defined in a certain domain G, be differentiable, as a function of a complex variable, at the point z of this domain, it is necessary and sufficient that the functions* $u(x, y)$ *and* $v(x, y)$ *be differentiable, as functions of two real variables, at the same point, and further that the conditions*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{2}$$

be satisfied.

When all the conditions of the theorem are fulfilled, the derivative $f'(z)$ *can be represented by any one of the following expressions :*

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \tag{3}$$

The conditions (2) are of a fundamental significance in the theory of analytic functions and also in its applications to problems of mechanics and physics. These are called *Cauchy-Riemann conditions (or equations).*

It should be noted here that this nomenclature, though generally accepted in texts and in scientific literature, is misleading from the historical point of view, because the conditions (2) had already been studied as far back as in the eighteenth century by D'Alembert and especially by Euler in connection with applications of the functions of a complex variable in hydromechanics (D'Alembert and Euler), in cartography and in integral calculus (Euler). Hence, it would be preferable to change the established terminology and refer to equations (2) as *D'Alembert-Euler equations.*

We now proceed to prove the foregoing theorem ; first, it will be shown that its conditions are necessary for the differentiability of the function $f(z)$.

In fact, if $f(z)$ is differentiable at the point z of the domain G, then

$$\Delta f(z) = f'(z)\,\Delta z + \varepsilon\,\Delta z, \tag{4}$$

where

$$\Delta z = z_1 - z = (x_1 - x) + i(y_1 - y) = \Delta x + i\,\Delta y,$$

$$\Delta f(z) = f(z_1) - f(z) = [u(x_1, y_1) - u(x, y)] + i[v(x_1, y_1) - v(x, y)] = \Delta u + i\,\Delta v,$$

$$f'(z) = a + ib, \quad \varepsilon = \varepsilon_1 + i\varepsilon_2 ;$$

ε_1 and ε_2 tend to zero as Δx and Δy go simultaneously to zero.

Separating the real and imaginary parts of (4), we have

$$\Delta u = a\,\Delta x - b\,\Delta y + \varepsilon_1\,\Delta x - \varepsilon_2\,\Delta y,$$
$$\Delta v = b\,\Delta x + a\,\Delta y + \varepsilon_2\,\Delta x - \varepsilon_1\,\Delta y.$$

Hence, as $\lim\limits_{\Delta x,\,\Delta y\to 0} \varepsilon_1 = \lim\limits_{\Delta x,\,\Delta y\to 0} \varepsilon_2 = 0$, it follows that

1. the functions $u(x, y)$ and $v(x, y)$ of the two real variables x and y are differentiable at the point (x, y);

2. their partial derivatives at this point are given by

$$\frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = -b, \quad \frac{\partial v}{\partial x} = b \text{ and } \frac{\partial v}{\partial y} = a$$

and, therefore, satisfy the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Finally, we obtain

$$f'(z) = a + ib = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

Thus, the necessity of the conditions of the theorem is proved.

We shall now prove that the conditions of the theorems are also sufficient. Let us suppose that these conditions are fulfilled. Then

$$\left.\begin{aligned} \Delta u &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \alpha_1\,\Delta x + \alpha_2\,\Delta y, \\ \Delta v &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \beta_1\,\Delta x + \beta_2\,\Delta y, \end{aligned}\right\} \tag{5}$$

where α_1, α_2, β_1 and β_2 tend to zero as Δx and $\Delta y \to 0$. Further,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = a, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = b. \tag{6}$$

Hence

$$\Delta u = a\,\Delta x - b\,\Delta y + \alpha_1\,\Delta x + \alpha_2\,\Delta y,$$
$$\Delta v = b\,\Delta x + a\,\Delta y + \beta_1\,\Delta x + \beta_2\,\Delta y,$$

and

$$\begin{aligned} \Delta f(z) = \Delta u + i\,\Delta v &= a(\Delta x + i\,\Delta y) + ib(\Delta x + i\,\Delta y) + \\ &+ (\alpha_1 + i\beta_1)\,\Delta x + (\alpha_2 + i\beta_2)\,\Delta y = (a + ib)\,\Delta z + \\ &+ \left[(\alpha_1 + i\beta_1)\frac{\Delta x}{\Delta z} + (\alpha_2 + i\beta_2)\frac{\Delta y}{\Delta z}\right]\Delta z = A\,\Delta z + \varepsilon\,\Delta z. \end{aligned} \tag{7}$$

Since

$$|\varepsilon| = \left|(\alpha_1+i\beta_1)\frac{\Delta x}{\Delta z} + (\alpha_2+i\beta_2)\frac{\Delta y}{\Delta z}\right| \leqslant |\alpha_1+i\beta_1|\left|\frac{\Delta x}{\Delta z}\right| +$$
$$+ |\alpha_2+i\beta_2|\left|\frac{\Delta y}{\Delta z}\right| \leqslant |\alpha_1+i\beta_1| + |\alpha_2+i\beta_2| \leqslant |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2|,$$

therefore ε tends to zero along with α_1, β_1, α_2 and β_2 as $\Delta z = \Delta x + i\Delta y$ tends to zero. Hence, it follows from (7) that the function $f(z)$ is differentiable and its derivative $f'(z)$ is equal to A:

$$f'(z) = A = a + bi = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \ldots .$$

This completes the proof.

It is well known from mathematical analysis that the existence and continuity of the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ of the functions $u(x, y)$ and $v(x, y)$ is sufficient for the differentiability of these functions. Hence, for the differentiability of the function $f(z) = u + iv$ it is sufficient that the derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ should exist, be continuous and satisfy the equations (2).

The function $f(z)$, differentiable at every point of the domain G, is called differentiable in this domain and also *holomorphic* or *analytic* (often referred to as *regular* or *proper*). The term holomorphic derived from the Greek ('oloc'—, *all whole*, and 'morphos'—*form, kind*) was introduced by Cauchy, Briot and Bouquet. The latter authors wrote, "By this term we wish to indicate that this (holomorphic) function is similar to the integral functions (that is, polynomials), possessing the same properties in the whole plane". The meaning of the term 'analytic', used earlier by Lagrange and then by Weierstrass, and in general use now-a-days, has been explained in the Introduction. The applicability of this term to functions of a complex variable, differentiable in a certain domain, will be justified later when we show that such a function may be represented, in a certain neighbourhood of any point of the domain, in the form of a convergent power series. For the present, we shall use the term 'analytic function' as a synonym with the term 'function of a complex variable, differentiable in a given domain'.

As an illustration let us examine the function $f(z) = e^x(\cos y + i\sin y)$, defined on the whole of the plane. Here

$$u = e^x \cos y, \quad v = e^x \sin y,$$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Thus the conditions (2) are satisfied ; hence, the function $f(z)$ is analytic in the whole plane. For its derivative, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = f(z).$$

In the example considered at the end of Art. 5, $f(z) = x$, $u = x$, $v = 0$, $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$, and the D'Alembert-Euler conditions are not fulfilled : $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$. We have already seen that this function is nowhere differentiable with respect to the z-plane.

In many cases it is convenient to express the conditions of differentiability of a function of a complex variable $f(z) = u + iv$, at $z \neq 0$, in terms of the polar coordinates : $|z| = r$ and $\operatorname{Arg} z = \Phi$. These (necessary and sufficient) conditions are :

1'. that u and v are differentiable functions of r and Φ ;

2'. that their partial derivatives are connected by the relations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \Phi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \Phi}. \tag{8}$$

To demonstrate this, it is sufficient to show that u and v are differentiable, as functions of r and Φ ($r \neq 0$), if, and only if, they are differentiable as functions of x and y, and that under these conditions equations (8) are equivalent to equations (2). The fulfilment of the first requirement follows from the fact, known from analysis, that a differentiable function [say, $u = u(x, y)$] of differentiable functions (say, $x = r\cos\Phi$ and $y = r\sin\Phi$) is also differentiable (with respect to the variables r and Φ). The second condition can be directly verified. For example, if condition (1') and also conditions (2) are fulfilled, we obtain

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\Phi + \frac{\partial u}{\partial y}\sin\Phi = \frac{\partial v}{\partial y}\cos\Phi - \frac{\partial v}{\partial x}\sin\Phi = \frac{1}{r}\frac{\partial v}{\partial \Phi}, \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\Phi + \frac{\partial v}{\partial y}\sin\Phi = -\frac{\partial u}{\partial y}\cos\Phi + \frac{\partial u}{\partial x}\sin\Phi = -\frac{1}{r}\frac{\partial u}{\partial \Phi}. \end{aligned}\right\} \tag{9}$$

Conversly, the reader can also arrive at the relations (2) from the conditions (8).

Writing equations (9) in the form

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\Phi - \frac{\partial v}{\partial x}\sin\Phi,$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\Phi + \frac{\partial u}{\partial x}\sin\Phi,$$

we can readily obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\Phi + \frac{\partial v}{\partial r}\sin\Phi, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial r}\sin\Phi + \frac{\partial v}{\partial r}\cos\Phi,$$

and, consequently,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial r}(\cos\Phi - i\sin\Phi) + i\frac{\partial v}{\partial r}(\cos\Phi - i\sin\Phi) =$$
$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\Phi - i\sin\Phi) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right). \tag{10}$$

This formula is convenient for calculating $f'(z)$ in terms of the polar coordinates. Further, equations (8) enable us to write $f'(z)$ also in the form

$$f'(z) = \frac{1}{z}\left(\frac{\partial v}{\partial\Phi} - i\frac{\partial u}{\partial\Phi}\right). \tag{11}$$

As an example we consider the function

$$z^{\frac{m}{n}} = |z|^{\frac{m}{n}}\left(\cos\frac{m\ \mathrm{Arg}\ z}{n} + i\sin\frac{m\ \mathrm{Arg}\ z}{n}\right) = r^{\frac{m}{n}}\left(\cos\frac{m\Phi}{n} + i\sin\frac{m\Phi}{n}\right),$$

where m is an integer and n a natural number. This function is defined in the domain $G: z \neq 0$, and is many-valued if the rational number m/n is not integral (*cf.* Chapter 1, Art. 2). The many-valuedness of the function $z^{m/n}$ is due to the many-valuedness of the argument Φ. In order that we can possibly speak of the derivative of this many-valued function at a certain point z of the domain G, we take up in this domain a neighbourhood of the point z not containing the origin of the coordinates. Then, fixing one of the values of Φ at the point z, we take all other points z_1 of the same neighbourhood such that the values of Φ_1 satisfy the condition $|\Phi_1 - \Phi| < \frac{\pi}{2}$ (see Fig. 5). We shall thus get a *one-valued* and *continuous* branch of the function $z^{m/n}$ in the given neighbourhood; let us denote this one-valued function by that very symbol $f(z) = z^{m/n}$.

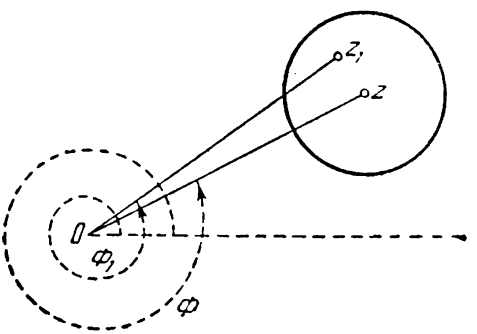


Fig. 5.

Obviously, in this case,

$$u = r^{\frac{m}{n}}\cos\frac{m\Phi}{n}, \qquad v = r^{\frac{m}{n}}\sin\frac{m\Phi}{n}.$$

$$\frac{\partial u}{\partial r} = \frac{m}{n}r^{\frac{m}{n}-1}\cos\frac{m\Phi}{n} = \frac{1}{r}\frac{\partial v}{\partial\Phi},$$

$$\frac{\partial v}{\partial r} = \frac{m}{n}r^{\frac{m}{n}-1}\sin\frac{m\Phi}{n} = -\frac{1}{r}\frac{\partial u}{\partial\Phi},$$

and, hence, $f(z)$ is a differentiable function. We get, from the formulae (10), its derivative in the form

$$f'(z) = \frac{r}{z}\left(\frac{m}{n} r^{\frac{m}{n}-1} \cos\frac{m\Phi}{n} + \frac{m}{n} r^{\frac{m}{n}-1} \sin\frac{m\Phi}{n}\right) =$$

$$= \frac{m}{n} r^{\frac{m}{n}}\left(\cos\frac{m\Phi}{n} + \sin\frac{m\Phi}{n}\right)\frac{1}{z} = \frac{m}{n}\frac{f(z)}{z}.$$

Thus, the reader will find that the rule of differentiation of the 'fractional power' $z^{m/n}$ remains formally the same as that for the corresponding function $x^{m/n}$ of a real variable. It is necessary to recall that our calculation has been carried out under the assumption $z \neq 0$, which can be omitted only when m/n is a nonnegative integer.

As an exercise for the reader we leave it to be proved that the function $f(z) = \ln r + i\Phi$, defined in the same domain G, is differentiable and that its derivative is $1/z$ (here, it is also necessary to separate out the one-valued and continuous branches of the function).

8. GEOMETRICAL INTERPRETATION OF THE ARGUMENT OF A DERIVATIVE

First of all we examine the complex function $z = \lambda(t)$ of the real variable t, defined and continuous on a certain segment of E, say $[\alpha, \beta]$ of the real axis. As was pointed out in Art. 4, a function like this defines a continuous curve L. We assume that there exists a derivative (relative to the set E) $\lambda'(t) \neq 0$ at some point of the segment $[\alpha, \beta]$. We shall show that in this case there exists a tangent T to the curve L at the point $z_0 = \lambda(t_0)$, (the tangent being understood to be the limiting position of the secants passing through the point z_0), the angle between T and the real axis being equal to $\operatorname{Arg}\lambda'(t_0)$.

As a matter of fact, we draw a secant through the points $z_0 = \lambda(t_0)$ and $z_1 = \lambda(t_1)$ of the curve L. We may assume that these two points do not coincide for all values of t_1 different from t_0 and sufficiently close to t_0 (otherwise, we find that the sequence $\{t_{1n}\} \to t_0$, such that

$$\lambda(t_{1n}) - \lambda(t_0) = 0$$

for all values of n. Consequently,

$$\lambda'(t_0) = \lim_{t_{1n} \to t_0} \frac{\lambda(t_{1n}) - \lambda(t_0)}{t_{1n} - t_0} = 0\Big).$$

Noting that the direction of the secant has the same sense as that of the vector $\frac{z_1-z_0}{t_1-t_0}$, we conclude that the secant tends to a limit as $t_1 \to t_0$ ($z_1 \to z_0$) only if the angle between the foregoing vector and the real axis, that is, $\operatorname{Arg} \frac{z_1-z_0}{t_1-t_0}$, tends to a limit as $t_1 \to t_0$. But since, by virtue of the conditions assumed, there exists the limit

$$\lim_{t_1 \to t_0} \frac{z_1-z_0}{t_1-t_0} = \lambda'(t_0) \neq 0,$$

we have

$$\lim_{t_1 \to t_0} \operatorname{Arg} \frac{z_1-z_0}{t_1-t_0} = \operatorname{Arg} \lambda'(t_0).$$

This completes the proof.

Hence, *for a complex function of a real variable the existence of a nonvanishing derivative implies the existence of a tangent to the corresponding curve; the angle of inclination of the tangent to the real axis coincides with the argument of the derivative.*

We now turn to a function of a complex variable, $w = f(z)$, defined and continuous in a certain domain G. We assume that a derivative $f'(z_0) \neq 0$ exists at the point $z_0 \in G$. Now, through the point z_0 we draw a curve L given by $z = \lambda(t)$ ($\alpha \leqslant t \leqslant \beta$, $\lambda(\gamma) = z_0$) for which there exists the derivative $\lambda'(t_0) \neq 0$. By what has been said above the curve L has a tangent at the point $z_0 = \lambda(\gamma)$ with $\operatorname{Arg} \lambda'(t_0)$ as the angle of inclination. This curve is transformed by the mapping $w = f(z)$ into the curve Λ lying in the plane of w: $w = f[\lambda(t)] = \mu(t)$ ($\alpha \leqslant t \leqslant \beta$, $\mu(t_0) = f(z_0) = w_0$). Since, according to the rule of differentiation of composite functions, the function $\mu(t)$ is differentiable at the point $t = t_0$ and $\mu'(t_0) = f'(z_0)\lambda'(t_0) \neq 0$, the curve Λ will have a tangent at the point $w_0 = f(z_0)$ and the angle between the tangent and the real axis will be

$$\operatorname{Arg} \mu'(t_0) = \operatorname{Arg} [\lambda'(t_0) f'(z_0)] = \operatorname{Arg} \lambda'(t_0) + \operatorname{Arg} f'(z_0).$$

It therefore follows that when the curve L passes over to its image Λ the inclination of the tangent at the initial point of the curve changes by an amount equal to

$$\operatorname{Arg} \mu'(t_0) - \operatorname{Arg} \lambda'(t_0) = \operatorname{Arg} f'(z_0),$$

which is independent of this curve. If, from the point z_0, emerge any two curves L_1 and L_2 having tangents T_1 and T_2 at the point z_0, the tangents τ_1 and τ_2 to their images Λ_1 and Λ_2 at the point $w_0 = f(z_0)$ are obtained from T_1 and T_2 by rotating them through one and the same angle

$\operatorname{Arg} f'(z_0)$; hence, the angle between the curves L_1 and L_2 will be equal (in magnitude and sense) to the angle between Λ_1 and Λ_2. Thus, through mapping with the help of the continuous function $w = f(z)$, possessing a nonvanishing derivative $f'(z_0)$, all curves of the z-plane passing through the point z_0 and having tangents at this point are mapped onto curves in the w-plane passing through the point $w_0 = f(z_0)$ and also having tangents at this point, the angles between the curves being preserved by this mapping. Mapping by a continuous function preserving the angles between the curves, passing through a given point, is said to be *conformal at that point.*

In this case, if not only the magnitudes of the angles but also their directions are preserved (as was found in the case of the mapping discussed above) we have a *conformal mapping of the first kind.* If, however, the directions of the angles get changed into their opposite ones, then the *conformal mapping is said to be of the second kind.*

Hence, *mapping by a function of a complex variable which is analytic in a certain domain G is a conformal mapping of the first kind at all points at which the derivative is nonvanishing.* If the mapping is conformal at all points of the domain G, it is called a *conformal mapping* of the domain G.

General examples of a conformal mapping of the second kind are provided by the mappings effected by functions conjugate to analytic functions, viz. $w = \overline{f(z)}$ [assuming that $f'(z) \neq 0$].

We leave it to the reader to prove that if the derivative at any point vanishes, then the angles might be changed as well as preserved [examine, for instance, the mappings

$$f_1(z) = r^2(\cos \Phi + i \sin \Phi)$$

and

$$f_2(z) = r^2(\cos 2\Phi + i \sin 2\Phi) = z^2$$

at the point $z = 0$].

9. GEOMETRICAL INTERPRETATION OF THE MODULUS OF A DERIVATIVE

It has been shown in the previous section that $\operatorname{Arg} f'(z_0)$ expresses the angle of rotation of the tangent to the curve L at a certain point z_0 on it when the tangent goes over to that of the image Λ of the curve at the point $w_0 = f(z_0)$. In particular, if $f'(z_0)$ is a real positive number, the tangents to L at z_0 and to Λ at $f(z_0)$ are parallel (and directed in one and the same sense).

We shall now give the geometrical interpretation of the modulus $|f'(z_0)|$. For this purpose, we note that

$$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

and that the numbers $|z-z_0|$ and $|f(z)-f(z_0)|$ represent respectively the distances between the points z and z_0 in the plane of z and between their images $f(z)$ and $f(z_0)$ in the plane of w. If the ratio $\frac{|f(z)-f(z_0)|}{|z-z_0|}$ can be regarded as the 'coefficient of elongation' of the vector $z-z_0$ as a result of mapping by the function $w=f(z)$ (this sort of an elongation may be less than, equal to or greater than unity), then *the modulus of the derivative, $|f'(z_0)|$, may be regarded as a measure of the elongation caused at the point z_0 due to mapping by the function $w=f(z)$.* It follows from what has been said above that the elongation at the point z_0 does not depend on the choice of the vector $z-z_0$ emanating from this point. However, the modulus of the derivative does not coincide with the actual elongation of the vector $z-z_0$ but represents only the limit of this elongation as z tends to z_0.

10. EXAMPLE: LINEAR AND BILINEAR FUNCTIONS

As an illustration of the above, we examine a *bilinear function* $L(z)=\frac{az+b}{cz+d}$ (at least one of the number c and d being nonzero). First, let $c=0$. Then, $L(z)$ can be written in the form $L(z)=\alpha z+\beta\left(\alpha=\frac{a}{d},\ \beta=\frac{b}{d}\right)$. This is an *(entire) linear function.* It is defined for all values of z and has a derivative $L'(z)=\alpha$, which has a constant nonzero magnitude, if $\alpha\neq 0$. Hence, the function $L(z)$ brings about a conformal mapping of the whole plane of the complex variable z. In this mapping, tangents to all curves in the z-plane get rotated by one and the same angle $\operatorname{Arg}\alpha$, and the coefficient of elongation at all points is found to be $|\alpha|$. If $\alpha=1$, then $\operatorname{Arg}\alpha=2k\pi$, $|\alpha|=1$ and both rotation and elongation are absent. Since the mapping here takes the form $w=z+\beta$, so it reduces, evidently, to a shift of the plane as a whole through a vector β. If, however, $\alpha\neq 1$ (and $\alpha\neq 0$), the mapping may be represented by the form $w-\gamma=\alpha(z-\gamma)$, where γ is determined by the equation $\gamma=\alpha\gamma+\beta$. It thus follows that every vector $z-\gamma$ starting from the point γ gets rotated, due to the mapping, through an angle $\operatorname{Arg}\alpha$ and is elongated $|\alpha|$ times, thus converting into the vector $w-\gamma$ starting from the same point γ. This means that the mapping $L(z)=\alpha z+\beta$, when $\alpha\neq 1$ (and $\alpha\neq 0$), reduces to a rotation of the plane as a whole about the point $\gamma=\frac{\beta}{1-\alpha}$ through the angle $\operatorname{Arg}\alpha$ and to an elongation by the factor $|\alpha|$ relative to this point. Evidently, this mapping is a similarity transformation with its centre at the point $\gamma=\frac{\beta}{1-\alpha}$ and coefficient of similitude $|\alpha|$, accompanied by a rotation about the same point through

an angle Arg α. This is a conformal mapping in the simplest case.

Now let $c \neq 0$. Then, for $z \neq \delta = -d/c$, there exists the derivative

$$L'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{ad-bc}{c^2} \cdot \frac{1}{(z-\delta)^2}.$$

If the determinant $ad - bc \neq 0$ [$ad-bc = 0$ implies that $\frac{a}{c} = \frac{b}{d} = \lambda$ whence $a = c\lambda, b = d\lambda$ and $L(z) = \frac{az+b}{cz+d} = \frac{c\lambda z+d\lambda}{cz+d} \equiv \lambda$], then $L'(z) \neq 0$ for all $z \neq \delta$. Hence, the mapping $w = L(z)$ is conformal at all finite points different from δ. In this mapping, tangents to the curves passing through any point $z \neq \delta$ get rotated through an angle

$$\operatorname{Arg} L'(z) = \operatorname{Arg} \frac{ad-bc}{c^2} - 2 \operatorname{Arg} (z-\delta).$$

The angle of rotation of the tangents evidently changes from point to point and has one and the same value at those points for which $\operatorname{Arg}(z-\delta)$ is the same, that is, for all points on any rectilinear ray emanating from the point δ. The elongation at the point z in the case of the given mapping is $|L'(z)| = \left|\frac{ad-bc}{c^2}\right| \div |z-\delta|^2$ and also changes from point to point. It has one and the same value for those points for which $|z-\delta|$ has the same value, that is, for all points on a circle with its centre at the point δ. This elongation, in particular, is equal to unity at every point of the circle γ: $|z-\delta| = \frac{1}{|c|}\sqrt{|ad-bc|}$ (*the isometric circle of the bilinear transformation*) is greater than unity inside γ, tending to infinity when z tends to δ and is less than unity outside γ, tending to zero as z tends to infinity (Fig. 6).

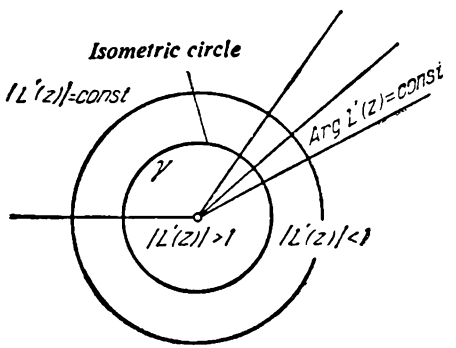


Fig. 6.

11. THE ANGLE FORMED AT THE POINT AT INFINITY

Let, as before, $c \neq 0$, and $ad - bc \neq 0$. Then, obviously,

$$\lim_{z \to \delta = -\frac{d}{c}} \frac{az+b}{cz+d} = \infty \quad \text{and} \quad \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} = \alpha.$$

We complete the definition of the function $L(z)$ by putting

$$L(\delta) = \infty \quad \text{and} \quad L(\infty) = \alpha.$$

Now the function $w = L(z)$ is defined on the whole of the extended plane of z, transforming the finite point δ of the z-plane into the point at infinity of the w-plane and the point at infinity of the z-plane into a finite point α of the w-plane.

From the equation $w = \dfrac{az+b}{cz+d}$, we obtain for the inverse function $z = L^{-1}(w)$ the expression

$$z = \frac{-dw+b}{cw-a}.$$

Here, we first assume that $w \neq \infty$ and $w \neq \alpha$ (then $z \neq \delta$ and $z \neq \infty$). For $w = \infty$ and $w = \alpha$, we would have the values $L^{-1}(\infty) = \delta$ and $L^{-1}(\alpha) = \infty$.

So, the function inverse to a bilinear one is itself bilinear. Here, we find that the bilinear function $w = L(z)$ effects a one-to-one mapping of the extended plane onto itself and also that this mapping is conformal for $z \neq \delta$ and $z \neq \infty$. In order that we can speak of a conformal mapping at these points we have to give an expedient definition of an angle with its vertex at the point at infinity. Let C_1 and C_2 be two continuous curves passing through the origin of coordinates where they form an angle θ. We map the plane onto itself by the transformation $\zeta = \dfrac{1}{z}$. Then, C_1 and C_2 are mapped onto continuous (in the generalized sense) curves C_1' and C_2' passing through the point at infinity. We shall say by definition that C_1' and C_2' form the same angle θ at the point at infinity. Thus, for example, the real and the imaginary axes form among themselves the angle $\pi/2$ at the origin of coordinates. By the mapping $\zeta = \dfrac{1}{z}$ each of them is mapped onto itself and the point $z = 0$ onto the point $\zeta = \infty$. It thus follows that the real and the imaginary axes intersect at the point at infinity at the same angle $\pi/2$ at which they intersect at the origin of coordinates.

We can say, in general, that *any two curves C_1' and C_2' passing through the point $z = \infty$ form there an angle θ if the images C_1 and C_2 of these curves under the mapping $\zeta = \dfrac{1}{z}$ form an angle θ at the origin of coordinates.*

We now turn to the general mapping

$$w = L(z) = \frac{az+b}{cz+d} \quad (ad-bc \neq 0, \ c \neq 0).$$

Let two curves C_1 and C_2 form an angle 0 at the point $\delta = -\frac{d}{c}$. Since $L(\delta) = \infty$, the images of these curves, $C_1' = L(C_1)$ and $C_2' = L(C_2)$, pass through the point $w = \infty$. By definition, the angle between them at ∞ must be equal to the angle formed previously, with vertex at the origin of coordinates, between the curves C_1'' and C_2'', obtained by the mapping $\zeta = \frac{1}{w}$. Evidently, C_1'' and C_2'' are images of the curves C_1 and C_2 obtained by the mapping $\zeta = \frac{1}{w} = \frac{cz+d}{az+b}$. But the latter is bilinear and, hence, conformal at the point $z = \delta$ (which it transforms into the finite point $\zeta = 0$). Hence, the curves C_1'' and C_2'' form an angle 0 at the point $\zeta = 0$, that is, the curves C_1' and C_2' form that very angle 0 at the point $w = \infty$. Thus, we have established the conformality of the mapping $w = L(z)$ at the point δ where the bilinear function becomes infinite. This result is also valid for the inverse function $z = L^{-1}(w)$ at the point α where this function becomes infinite. In other words, the mapping $w = L(z)$ is also conformal at the point at infinity (which it transforms into the point α). Finally, we find that *every bilinear function* $w = \frac{az+b}{cz+d}$ $(ad-bc \neq 0,\ c \neq 0)$ *maps the extended plane one-to-one and conformally onto itself.*

The reader may easily verify that this statement is also true for the (entire) linear function $w = az + b$. The only difference from the previous case is that here the point at infinity gets transformed into itself.

12. HARMONIC AND CONJUGATE HARMONIC FUNCTIONS

It will be shown later (Art. 3 of Chapter 6) that the real and the imaginary parts $u(x, y)$ and $v(x, y)$ of the function $f(z)$, analytic in a certain domain, possess differentiable (and hence continuous) partial derivatives of any finite order. In this section we shall derive certain consequences of this fact. Differentiating the first of the equations (.) with respect to x and the second with respect to y and observing that $\frac{\partial^2 v}{\partial y\, \partial x} = \frac{\partial^2 v}{\partial x\, \partial y}$, we obtain, by adding the corresponding terms of the two results, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A similar equation is also obtained for the function $v(x, y)$ if the first of the equations (2) is differentiated with respect to y, the second with respect to x and then the second result is subtracted term by term from

the first. One gets

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

The equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{12}$$

is a differential equation with partial derivatives of the second order. This is called the *Laplace equation*, and the functions possessing, in some domain, continuous partial derivatives upto the second order inclusive and satisfying this equation are called *harmonic functions*.

Hence, we can say that *the real and the imaginary parts of a function of a complex variable, analytic in some domain, are harmonic functions in this domain.* Thus, for example, the real part $x^2 - y^2$ and the imaginary part $2xy$ of the function $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, which is analytic in the whole z-plane, are harmonic in the whole plane. It has been shown in Art. 7 that the function $f(z) = \ln |z| + i \operatorname{Arg} z$ is analytic in the domain $G: z \neq 0$; hence, $\ln |z|$ and $\operatorname{Arg} z$ are harmonic functions in this domain. (This is easy to prove in view of the fact that $\ln |z| = \frac{1}{2} \ln (x^2 + y^2)$ and $\operatorname{Arg} z = \arctan \frac{y}{x} + C$ if $x \neq 0$, or $\operatorname{Arg} z = C - \arctan \frac{x}{y}$ if $y \neq 0$). We note that $\operatorname{Arg} z$ gives the simplest example of a many-valued harmonic function. Strictly speaking, the definition of harmonicity should be applied only to one-valued continuous branches of this function.

Let $\varphi(x, y)$ be a one-valued function which is harmonic in a given simply-connected domain G (for example, in a circle, in a half-plane or in the whole plane). We shall now show how to find the function $f(z) = u(x, y) + iv(x, y)$ which is analytic in this domain and whose real part coincides with $\varphi(x, y)$: $u(x, y) = \varphi(x, y)$. To find the imaginary part we have two equations (D'Alembert-Euler equations)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = P(x, y) \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = Q(x, y).$$

Functions $P(x, y)$ and $Q(x, y)$ are continuous in the domain G and possess in it continuous partial derivatives of the first order [these derivatives are expressed in terms of the second order partial derivatives of $u(x, y)$].

Here the condition

$$\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial Q}{\partial x}$$

is satisfied; consequently, the curvilinear integral

$$\int_{(x_0, y_0)}^{(x, y)} (P\,dx + Q\,dy)$$

does not depend on the form of the path connecting the points (x_0, y_0) and (x, y) of the domain G, and hence it represents the one-valued function $\psi(x, y)$ of the point (x, y).[1] This function has the same partial derivatives as the unknown function $v(x, y)$ has:

$$\frac{\partial \psi}{\partial x} = P = \frac{\partial v}{\partial x}, \quad \frac{\partial \psi}{\partial y} = Q = \frac{\partial v}{\partial y},$$

and hence $v(x, y)$ can differ from $\psi(x, y)$ only by a constant term:

$$v(x, y) = \psi(x, y) + C = \int_{(x_0, y_0)}^{(x, y)} \left[P\,dx + Q\,dy\right] + C =$$

$$= \int_{(x_0, y_0)}^{(x, y)} \left[-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right] + C;$$

here, C is a real number. Calculating v by this formula, we shall have two functions (differentiable in the domain G)

$$u = \varphi(x, y) \quad \text{and} \quad v = \psi(x, y) + C,$$

which are connected by the D'Alembert-Euler equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, it follows that the function $f(z) = u(x, y) + iv(x, y) = \varphi(x, y) + i\psi(x, y) + iC$ is analytic in the domain G. Thus, *from the function $\varphi(x, y)$, harmonic in a simply-connected domain G, we can derive an analytic function $f(z)$, the real part of which coincides with $\varphi(x, y)$. This function is determined to the extent of a purely imaginary constant.*

Example. Suppose G is obtained from the plane by excluding from it the semiaxis: $y = 0$, $x \leqslant 0$. It is easy to verify that the function

[1] See in this connection G. M. Fichtenholz, *Principles of Mathematical Analysis*, Vol. II, Gostekhizdat, Moscow, 1956, Chapter 21, Art. 4.

$u(x, y) = \frac{1}{2} \ln (x^2+y^2)$ is harmonic in this domain. For determining the imaginary part we have the equations :

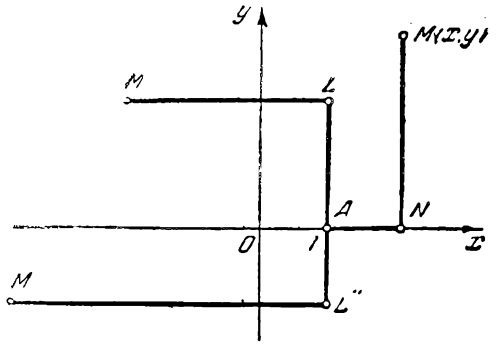


Fig. 7.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2},$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2},$$

and hence

$$v(x, y) = \int_{(1, 0)}^{(x, y)} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + C.$$

Now, if the point $M(x, y)$, say, lies in the right-hand half-plane ($x>0$) then by integrating along the two-section broken line ANM (Fig. 7), we have

$$v(x, y) = \int_0^y \frac{x\,dy}{x^2+y^2} + C = \arctan\frac{y}{x} + C.$$

If the point $M(x, y)$ lies in the second (or third) quadrant ($x \leqslant 0$, $y \neq 0$), then by integrating along the broken line $AL'M$ (or $AL''M$, respectively) we obtain

$$v(x, y) = \int_0^y \frac{dy}{1+y^2} - \int_1^x \frac{y\,dx}{x^2+y^2} + C = \arctan y - \arctan\frac{x}{y} + \arctan\frac{1}{y} + C.$$

Hence, it is obvious that $v(x, y) = \arg z + C$ at all points of G; consequently,

$$f(z) = \frac{1}{2} \ln (x^2+y^2) + i \arg z + iC = \ln |z| + i \arg z + iC.$$

The foregoing method for finding an analytic function from a knowledge of its real part is also applicable to the case of a multiply-connected domain G, but in this case the integral $\psi(x, y) = \int_{(x_0, y_0)}^{(x, y)} (P\,dx + Q\,dy)$ represents, when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, a many-valued function of the point (x, y). Hence, in general, the function $f(z) = \varphi(x, y) + i\psi(x, y) + iC$ is also found to be many-valued (though $\varphi(x, y)$ is one-valued as before). All this can be easily illustrated by finding an analytic

function $f(z)$ in the domain G, $z \neq 0$, from the given real part $\varphi(x, y) = \frac{1}{2} \ln (x^2+y^2)$. We then obtain the many-valued analytic function $f(z) = \ln |z| + i \operatorname{Arg} z$.

We finally note that exactly in the same way one can find an analytic function from a knowledge of its imaginary part. Obviously, if $\psi(x, y)$ is the imaginary part of $f(z)$ it would be the real part of $\frac{1}{i} f(z)$.

We come across harmonic functions in many problems of physics and mechanics. Thus, for instance, the temperature of a homogeneous plate in thermal equilibrium, the electric potential of a plane conductor, the velocity potential of the steady plane flow of a homogeneous incompressible fluid, etc. are harmonic functions of the Cartesian coordinates x and y in their respective domains. The utility of the theory of analytic functions for solving many a problem in physics and mechanics lies in that instead of finding the harmonic functions and operating with them we find analytic functions, the real and imaginary parts of which are represented by the respective harmonic functions.

13. HYDROMECHANICAL INTERPRETATION OF AN ANALYTIC FUNCTION

We shall now examine the *steady motion, parallel to a plane, of an incompressible homogeneous fluid.* The characteristic of this motion is that the velocity of every particle of the fluid is represented by a vector parallel to one and the same plane (x, y) and depending only on the coordinates x and y of the projection of the particle on this plane (that is, not depending either on the third coordinate ζ or on time). In this case it is sufficient to follow the motion only of the projections of the fluid particles on the plane (x, y), that is, to regard the motion as a *plane* motion. Accordingly, we shall speak of the motion of the fluid in the *plane* (x, y). Let G be the domain of the plane in which the fluid flows. The closed set F, complementary to G with respect to the plane, may be looked upon as the totality of projections of the cylindrical solid bodies past which the fluid flows in space. We call these projections simply *solid bodies past which the fluid flows.* In this scheme these bodies are stationary. However, the case of a rectilinear and uniform *forward* motion of a solid body (or of a system of solid bodies) in a fluid can also be reduced to the above scheme. For this purpose, it is sufficient, according to the classical Galilean principle, to ascribe to the total fluid a velocity constant in magnitude and direction and equal to the velocity of any point of the body. Then, the fluid will, instead of being at rest, have that very velocity at infinity; on the contrary, the bodies past which it flows can now be regarded as stationary.

Let $u(x, y)$ and $v(x, y)$ be the projections of the velocity vector of a fluid particle located at the point (x, y) [these functions are assumed to be continuous]. We examine any arc γ of a smooth curve connecting two points z_1 and z_2 of the domain G. If ds is an element of the arc γ and n the direction of the normal to ds, so drawn that n remains to the right of the arc γ when traversing it from the point z_1 to the point z_2, then the area of the parallelogram constructed from ds and the velocity vector $u + iv$ (as adjacent sides) is obviously equal to the product of ds with the projection of this vector upon the normal, that is

$$[u \cos \widehat{(n, x)} + v \cos \widehat{(n, y)}]\, ds. \tag{13}$$

This quantity has a '+' sign if the velocity vector makes an acute angle with n and has a '−' sign if it makes an obtuse angle with it. Evidently, the above mentioned parallelogram may be considered as the base of a rectangular parallelepiped with its height equal to unity (perpendicular to the xy-plane). The volume of this parallelepiped is equal, in absolute magnitude, to (13). This therefore represents, when taken with a definite sign, the volume of the fluid belonging to a layer with height equal to unity and parallel to the plane (x, y), passing per second through the small area projecting on to the arc element ds. The total volume of the fluid, belonging to the above mentioned layer and passing per second through the cylindrical area projecting on to the arc γ, will be

$$\int_{\gamma} [u \cos \widehat{(n, x)} + v \cos \widehat{(n, y)}]\, ds.$$

Noting that under the given conditions the angle $\widehat{n, x}$ exceeds by $3\pi/2$ the angle $\widehat{t, x}$ between the tangent to γ, drawn in the sense in which this curve is traversed, and the real axis, we find that

$$\cos \widehat{(n, x)} = \sin \widehat{(x, t)} = \frac{dy}{ds}.$$

Similarly

$$\cos \widehat{(n, y)} = -\cos \widehat{(x, t)} = -\frac{dx}{ds}.$$

Consequently, the above integral may be written in the form

$$\int_{\gamma} \left(u \frac{dy}{ds} - v \frac{dx}{ds}\right) ds = \int_{\gamma} (-v\, dx + u\, dy). \tag{14}$$

The quantity thus obtained is called the *flow* of the fluid across the curve γ. If the curve is closed and its positive direction is so chosen that the interior of the curve lies to the left of an observer who traverses along the curve in this direction, the normal n is directed towards the exterior of the curve γ. Hence, the flow across the element ds of the boundary is positive if the fluid flows from within γ and negative if it flows into the interior of γ. We assume that the interior of γ belongs to the domain G occupied by the moving fluid and that there are no *sources* in it from which the fluid might flow out nor any *sinks* where it might cease to flow further. Then, the total quantity of flow across γ must be equal to zero :

$$\int_{\gamma}(-v\,dx + u\,dy) = 0\,.$$

Extending this result to all closed curves of any simply-connected subdomain $g \subset G$, which contains neither any sources nor any sinks, we arrive at the conclusion that the flow across any arc belonging to this domain does not depend on the form of this arc ; it rather depends only on the choice of its ends z_1 and z_2.

We assume that u and v possess continuous partial derivatives. It then follows from the condition obtained above that[1]

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y}\,. \tag{15}$$

We have thus obtained the so-called *equation of continuity* for an incompressible fluid. Evidently, this is one of the D'Alembert-Euler equations for the pair of functions $u\,(x, y)$ and $-v\,(x, y)$. To arrive at the other D'Alembert-Euler equation we examine the integral

$$\int_{\gamma}(u\,dx + v\,dy)\,, \tag{16}$$

taken around a closed curve γ. The expression $(u\,dx+v\,dy)$ represents the projection of the velocity vector on the element ds of the arc γ (more precisely, the product of the projection of the velocity on the tangent drawn in the sense of the transversed curve and the length ds of the arc element). The integral (16) is called the *velocity circulation* along the curve γ. We assume that in a certain simply-connected subdomain $g \subset G$ the velocity circulation is equal to zero for any closed curve

[1] vide footnote to page 41.

belonging to G. Then, evidently, the condition

$$\frac{\partial u}{\partial y} = -\frac{\partial (-v)}{\partial x} \tag{17}$$

must be satisfied in the given domain. This is the second D'Alembert-Euler equation for the pair of functions u and $-v$. In terms of its physical meaning, the condition expresses the absence of vorticity in the motion of the fluid (in the subdomain $g \subset G$). In general, the vector perpendicular to the plane (x, y) and having a projection equal to $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ on the third coordinate axis ζ is called the *vortex of the velocity* $u + iv$ in the plane motion. The vortex of the velocity characterizes the rotational motion of the fluid particles. If we assume the fluid to get solidified, then its angular velocity of rotation at the point (x, y) would have the value $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$. Thus, the absence of the vortex in the given domain g implies that the particles have at every point of this domain only forward motion, and that the fluid is subjected to certain deformations, without undergoing rotation, which may be looked upon as a component of the motion at the given point. When there are no vortices, equation (17) is satisfied and, hence, the velocity circulation for any closed curve reduces to zero.

Assuming that the equation (15) [which presupposes the absence of sources and sinks] and the equation (17) [which presupposes the absence of vortices] are satisfied simultaneously in the given subdomain $g \subset G$, we find that the function

$$u + (-v)\, i = u - iv,$$

conjugate to the velocity of motion of the fluid particles, is an analytic function of the variable $z = x + iy$.

It will be shown later (Art. 7, Chapter 5) that every function which is analytic in a simply-connected domain of the finite plane may be looked upon as the derivative of a certain one-valued analytic function.

Since g is a simply-connected domain, $u - iv$ may be looked upon as the derivative of a one-valued analytic function $f(z)$ in this domain; this function can be determined accurately only to the extent of an additive constant.

This function, satisfying the condition

$$f'(z) = u - iv,$$

is called the *complex potential* or the *characteristic flow function.*

We obtain

$$f(z) = \varphi(x, y) + i\psi(x, y). \tag{18}$$

Then

$$f'(z) = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \varphi}{\partial y}$$

and, consequently,

$$\frac{\partial \varphi}{\partial x} = u, \quad \frac{\partial \varphi}{\partial y} = v,$$

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u.$$

The first pair of relations obtained above shows that $u(x, y)$—the real part of the complex potential—is the *velocity potential* of the motion in question. And for it we obviously have

$$\varphi(x, y) - \varphi(x_0, y_0) = \int_{(x_0, y_0)}^{(x, y)} (u\,dx + v\,dy).$$

It follows from the second pair of relations that

$$\psi(x, y) - \psi(x_0, y_0) = \int_{(x_0, y_0)}^{(x, y)} (-v\,dx + u\,dy),$$

that is, the difference between the values of the function $\psi(x, y)$ at two points is equal to the flow across any curve joining these points. The function $\psi(x, y)$—the imaginary part of the complex potential—is called the *flow function.*

We now examine two families of curves :

$$\varphi(x, y) = \text{const} \tag{19}$$

and

$$\psi(x, y) = \text{const}. \tag{20}$$

These are represented, in the plane of the values of the function $\zeta = f(z)$, by the families of coordinate straight lines $\xi = \text{const}$ and $\eta = \text{const}$. Since the latter are mutually orthogonal, the families of the curves (19) and (20) are also mutually orthogonal, by virtue of the conformality of

the mapping $\zeta = f(z)$. (This inference is valid only if $f'(z) \neq 0$, that is, if the velocity of the particle is nonvanishing).

The curves (19) are characterized by the relation

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = 0,$$

that is,

$$u\,dx + v\,dy = 0; \tag{21}$$

these are known as the *equipotential curves*. The relation

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0,$$

that is,

$$-v\,dx + u\,dy = 0$$

or

$$\frac{dx}{u} = \frac{dv}{v} \tag{22}$$

is characteristic for the curves (20). These are the so-called *flow lines*.

It follows from (21) that the velocity vector $u + iv$ at the points where it is nonvanishing is directed along the normal to the corresponding equipotential line. Similarly, it follows from (22) that this vector is directed along the tangent to the flow line. Hence, it again follows that the equipotential lines and the flow lines are mutually orthogonal.

Moreover, since the velocity vector is directed along the tangent to the flow line and the motion is steady (velocity depending only on the position of the particle), it follows that the flow lines coincide with the trajectories of the particles.[1]

If the domain G of the fluid flow contains separate (isolated) sources of flow, sinks or vortices (that is, the points to which nonvanishing vortices correspond), then a removal of these from the domain G give a multiply-connected domain G'. This also applies to every simply-connected subdomain of this domain. Hence, as before, it follows that the function $u - iv$, conjugate to the velocity $u + iv$, represents a one-valued analytic function in the whole of the domain G'.

The function (18) (generally many-valued) will, as before, be the complex potential of the flow, which will decompose into one-valued

[1] In the general case of unsteady motion, we can also talk of flow lines as the curves tangents to which coincide with the velocity vectors at a given instant of time. But here, the flow lines do not generally coincide with the trajectories of the particles.

analytic branches in every simply-connected subdomain $g' \subset G'$. Since the derivative of each of these coincides with one and the same function $u - iv$, the different branches of the complex potential may differ from one another only by a constant.

Thus, we have established that to every steady plane motion of an incompressible fluid in a certain domain G there corresponds a function $f(z)$—the complex potential of the motion, which is analytic at all points of the domain G, except at those having sources, sinks or vortices.

This function is, in general, many-valued but its derivative, representing at every point of the domain a complex number, conjugate to the velocity $u + iv$, is one-valued.

The totality of the profiles (projections or sections) of the walls of the vessel containing the fluid or of the cylindrical bodies past which it flows, may be treated as the boundary of the domain. Since the particles of the fluid immediately adjacent to the walls must slide along them, the boundary of the domain must also be included in the system of flow lines.

In general, if we have in a domain G a function $f(z)$ analytic everywhere, except at certain isolated points, and generally many-valued but having a one-valued derivative $f'(z)$, then it may be interpreted as the complex potential of a certain fluid flow in this domain. In that case, the points in the domain G at which the analyticity of the function $f(z)$ fails can in the simplest cases be interpreted as sources, sinks, or vortices of the flow and the boundary of the domain can be taken to be the profile of the solid bodies past which it flows. For the latter interpretation to be valid, it is necessary that the boundary of the domain G be included in the system of flow lines, that is, the flow function $[\psi(x, y) = \operatorname{Im} f(z)]$ should be constant for all curves, having a particular value for each curve.

14. EXAMPLES

For the purpose of illustrating the general arguments of the previous section we shall first examine the case of the integral linear function

$$f(z) = az.$$

This function can be looked upon as the complex potential of the motion of a fluid occupying the whole plane. This motion is steady and its velocity is $f'(z) = \bar{a}$ at every point. Putting $a = \alpha + i\beta$, we obtain for the velocity potential the expression

$$\varphi(x, y) = \alpha x - \beta y,$$

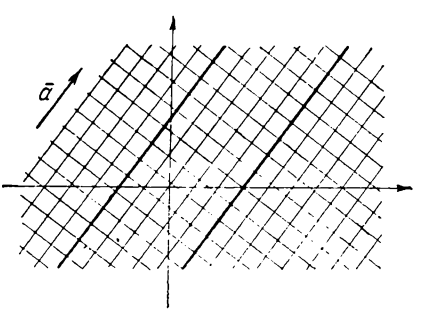


Fig. 8.

and for the flow function

$$\psi(x, y) = \beta x + \alpha y.$$

Flow lines, and equipotential lines which are orthogonal to the

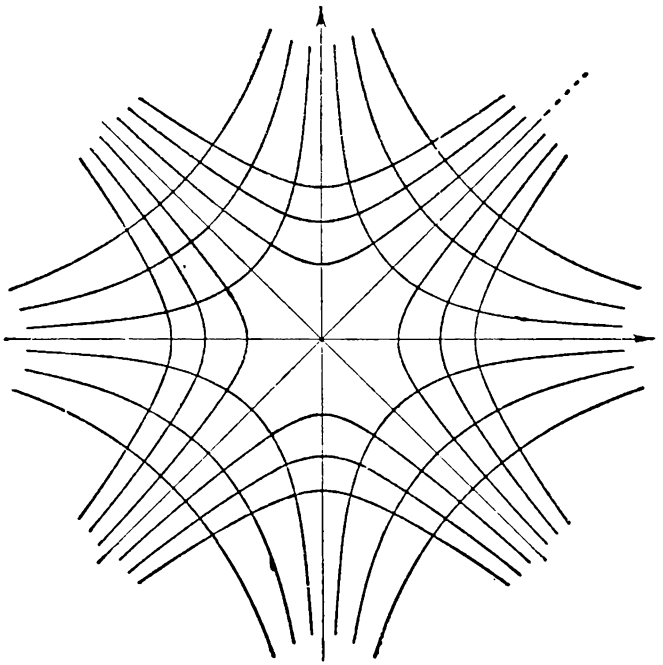


Fig. 9.

flow lines, have been drawn in Fig. 8. If instead of the whole plane we examine only the strip bounded by two straight lines, parallel to the vector $\bar{a}$, then the same function represents the complex potential of the flow of the fluid in the strip.

We take, as another example, the function

$$f(z) = z^2.$$

This too is the complex potential of the motion of a fluid occupying the whole plane. The velocity of a fluid particle located at the point z is equal to $\overline{f'(z)} = 2\bar{z}$, the velocity potential having the form

$$\varphi(x, y) = x^2 - y^2,$$

and the flow function

$$\psi(x, y) = 2xy.$$

Fig. 9 shows the equipotential lines, $x^2 - y^2 = \text{const}$, and the flow lines, $2xy = \text{const}$, of the flow in question. Evidently, these are rectangular hyperbolas. Both the coordinate axes, $2xy = 0$, pertain to the flow lines. Velocity at the point of intersection of these axes (the origin of coordinates) is zero. Examining one of the quadrants instead of the whole plane, say, the first quadrant, we find that this function represents the complex potential of the plane motion of a fluid enclosed in the first quadrant. Here, the two arms of the (right) angle represent the walls of the vessel in which the fluid moves.

3

ELEMENTARY ANALYTIC FUNCTIONS AND THE CORRESPONDING CONFORMAL MAPPINGS

1. THE POLYNOMIAL

The simplest and the most important class of differentiable functions consists of the one-valued functions, analytic in the entire plane with the point at infinity excluded from it. Such functions are called *entire* functions. A particularly specific example of an entire function is the *polynomial*

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = P_n(z).$$

This can also represent a constant ($n=0$). However, if $n > 0$ and $a_n \neq 0$, then $\lim_{z \to \infty} P_n(z) = \infty$. Hence, a polynomial of a degree higher than zero becomes ∞ at the point at infinity.

If w is any arbitrary (proper) complex number, then, as we know from algebra, the equation $P_n(z) = w$ has n roots, of which some may coincide (the so-called multiple roots). Hence, every point of the w-plane belongs to the image of the z-plane under the mapping $w = f(z)$ and is an image of n points: $z_1, z_2, \ldots, z_n$. We add that $P_n(\infty) = \infty$, and hence ∞ also belongs to the image of the extended plane. The point at infinity, $w = \infty$, will be the image of the root(s) of the equation $P_n(z) = \infty$ which would also be the point at infinity in the z-plane. We shall consider this ∞, for the purposes of symmetry, as an n-tuple root of this equation [$P_n(z) = \infty$]. Thus, *a polynomial of degree* n ($a_n \neq 0$, $n > 0$) *maps the extended plane onto itself in such a manner that every point* w *of the image has* n *pre-images* $z_1, z_2, \ldots, z_n$. However, as has already been pointed out, for certain distinct, exceptional values of w ($w = \infty$ is one of such values) the number of pre-images may be less than n.

It can be easily seen that the number of such exceptional values cannot exceed n. In fact, if $w_0 \neq \infty$ and the equation $P_n(z) = w_0$ has

multiple roots, then, as we know from algebra, for every one of them $P_n'(z) = 0$. But the last equation has $n-1$ roots (of which some may be coincident) $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. It, therefore, follows that w_0 must have one of the following $n-1$ values : $P_n(\zeta_1), \ldots, P_n(\zeta_{n-1})$, and if we further add to these the point at infinity, we shall obtain, at the most, those very n points of the w-plane which have less than n pre-images in the w-plane.

2. POINTS AT WHICH THE CONFORMALITY OF THE MAPPING FAILS

By the general theory the mapping $w = P_n(z)$ is conformal at all points. excepting those, namely $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$, at which the derivative vanishes and so also at the point $z = \infty$.

When $n = 1$, the polynomial is an entire linear function ; in this case, the mapping is one-to-one and conformal in the whole extended plane, including the point at infinity (*cf.* Arts. 10 and 11, Chapter 2). For $n > 1$, the conformality actually fails at these points.

Now, let $P_n'(z_0) = 0$. Then, $z = z_0$ is a multiple root of the equation $P_n(z) - P_n(z_0) = 0$ and, consequently, $P_n(z) - P_n(z_0)$ may be represented in the form

$$P_n(z) - P_n(z_0) = (z-z_0)^k Q(z),$$

where $k \geqslant 2$ is the multiplicity of the root $z = z_0$ (the number k, as we know, is greater by one than the multiplicity of the root $z = z_0$ of the equation $P_n'(z) = 0$; the polynomial $Q(z)$ does not vanish for $z = z_0$. Putting $P_n(z) = w$ and $P_n(z_0) = w_0$, we get

$$\operatorname{Arg}(w-w_0) = \operatorname{Arg}(z-z_0)^k + \operatorname{Arg} Q(z),$$

and further

$$\lim_{z \to z_0} \{\operatorname{Arg}(w-w_0) - \operatorname{Arg}(z-z_0)^k\} = \operatorname{Arg} Q(z_0).$$

Now, let $z = \lambda(t)$ be the curve L starting from the point z_0 $[z_0 = \lambda(t_0)]$ and having at this point a tangent inclined at an angle

$$\operatorname{Arg} \lambda'(t_0) = \lim_{t_1 \to t_0,\ t_1 > t_0} \operatorname{Arg} \frac{z_1 - z_0}{t_1 - t_0}$$

to the real axis (*cf.* Art. 8, Chapter 2). The image Λ of this curve in the w-plane will be : $w = P_n[\lambda(t)] = \mu(t)$. From this we cannot directly infer the existence of a tangent to Λ at the point w_0 $(t = t_0)$ [because $\mu'(t_0) = P_n'(z_0) \cdot \lambda'(t_0) = 0$]. But, for the slope of the secant passing through

the points w_0 and $w \neq w_0$ we have, for $t > t_0$, the expression

$$\operatorname{Arg} \frac{w-w_0}{t-t_0} = \operatorname{Arg} \frac{w-w_0}{(z-z_0)^k} + k \operatorname{Arg} \frac{(z-z_0)}{t-t_0} \to$$
$$\to \operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda'(t_0) \quad \text{for } t \to t_0.$$

It thereby follows that the tangent exists.

If L_1 and L_2 are two curves, $z = \lambda_1(t)$ and $z = \lambda_2(t)$, starting from the point z_0 and forming an angle 0 at this point such that

$$0 = \operatorname{Arg} \lambda_2'(t_2) - \operatorname{Arg} \lambda_1'(t_1) \quad [\lambda_1(t_1) = \lambda_2(t_2) = z_0],$$

then the images of these curves, Λ_1 and Λ_2, would start from the point w_0 and form there the angle

$$[\operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda_2'(t_2)] - [\operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda_1'(t_1)] =$$
$$= k [\operatorname{Arg} \lambda_2'(t_2) - \operatorname{Arg} \lambda_1'(t_1)] = k0.$$

Thus, *under the mapping* $w = P_n(z)$, *all angles with their vertices at the points at which the derivative* $P_n'(z)$ *vanishes undergo an increase by a factor equal to the multiplicity of the corresponding root of the equation* $P_n(z) - P_n(z_0) = 0$.

The reader can, by employing the transformation $\zeta = 1/z$, easily see that, for $n > 1$, the conformality of the mapping fails at the point at infinity also. The angles formed by curves at the point at infinity increase n times as a result of the mapping $w = P_n(z)$.

3. MAPPING OF THE FORM $w = (z-a)^n$

We examine, in particular, the mapping $w = (z-a)^n$ $(n>1)$. It maps the extended plane onto itself such that every point of the w-plane has n pre-images in the z-plane. The points $w = 0$ and $w = \infty$ are exceptions to this. For these points the pre-images merge into one point, a or ∞, respectively. The pre-images z $(\neq a, \neq \infty)$ are determined by the equation

$$w = (z-a)^n,$$

so that

$$z = a + \sqrt[n]{w} = a + \sqrt[n]{|w|} \cdot \cos\left(\frac{\operatorname{Arg} w}{n} + i \sin \frac{\operatorname{Arg} w}{n}\right).$$

Evidently, these n points are located at the vertices of a regular polygon with centre at a.

The mapping $w = (z-a)^n$ is conformal at all points except at $z = a$ and $z = \infty$. Angles with vertices at these two points get increased n times as a result of the given mapping.

For the sake of clarity, we further observe that

$$| w | = | z - a |^n \text{ and } \operatorname{Arg} w = n \operatorname{Arg} (z-a).$$

It then follows that a circle of radius r with its centre at the point $z = a$ is mapped onto a circle with radius r^n with its centre at the point $w = 0$. If in this case the point z traverses the circle $| z - a | = r$ once in the positive direction (that is, $\operatorname{Arg} (z-a)$, increasing continuously, gets enhanced by 2π), then the point w traverses the circle $| w | = r^n$ in the same direction n times (that is, $\operatorname{Arg} w$, increasing continuously, gets enhanced by $2\pi n$). We now make the point z traverse the rectilinear ray $\operatorname{Arg} (z-a) = \varphi_0 + 2k\pi$ from the point a to ∞. Our formula shows that the corresponding point w traverses the rectilinear ray $\operatorname{Arg} w = n\varphi_0 + 2m\pi$ from the origin of coordinates to infinity.

We now examine the domain g, comprising the interior of an angle of aperture θ, $0 < \theta \leqslant \frac{2\pi}{n}$, with its vertex at the point a. Let this angle be bounded by the rectilinear rays

$$\operatorname{Arg} (z-a) = \varphi_0 + 2k\pi$$

and

$$\operatorname{Arg} (z-a) = \varphi_1 + 2m\pi$$

$$(\varphi_1 - \varphi_0 = \theta).$$

It follows from the foregoing that the image of the domain g in the w-plane will be a domain d, represented by an angle of aperture

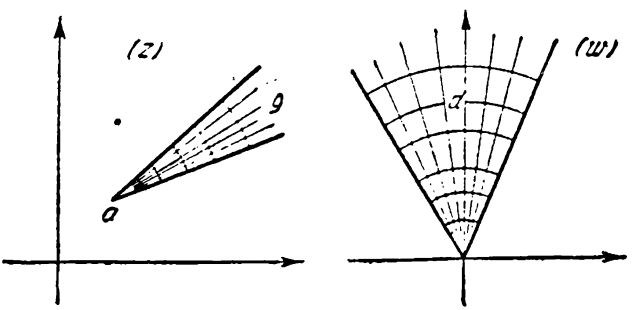


Fig. 10.

$n\theta$ with its vertex at the origin of coordinates, bounded by two rectilinear rays (Fig. 10). The correspondence between g and d,

established by the function $w = (z-a)^n$, will be one-to-one. In fact, since the function $w = (z-a)^n$ is one-valued, it will be sufficient, in order to prove this statement, to establish that every point w of the domain d has only one pre-image in the domain g. For this purpose we note that all the n pre-images of the point w are located in the z-plane at the vertices of a regular polygon with its centre at a, such that any two of them can fall in the interior of a certain angle with its vertex at a only if the aperture of the angle is greater than $2\pi/n$. But the aperture of the angle g is not greater than $2\pi/n$; hence, in the angle g, there exists only one pre-image of every point of d; this completes the proof of the statement under reference.

Thus, *the function* $w = (z-a)^n$ *maps in a one-to-one manner and conformally the interior of any angle with rectilinear arms, with its vertex at the point a and with an aperture* θ, $0 < \theta \leqslant 2\pi/n$, *onto the interior of a corresponding angle, also with rectilinear arms, having its vertex at the origin of coordinates and an aperture* $n\theta$.

Hence, whenever we have to map an angle with rectilinear arms onto another angle, a number of times larger than the former one, we take recourse to the foregoing.

It would, of course, be erroneous to think that under the mapping $w = (z-a)^n$ $(n>1)$ every straight line, in general, is transformed into a straight line and every circle into a circle. We take, say, $a = 0$ and $n = 2$. Then we obtain the function $w = z^2$. We shall examine the transformation of straight lines not passing through the origin and parallel to one of the coordinate axes, by the function $w = z^2$. We take, say, a straight line parallel to the imaginary axis : $z = c + it$, $c \neq 0$, $-\infty < t < +\infty$. As its image we obtain the line $w = (c+it)^2$ or, putting $w = u + iv$ and separating the real and imaginary parts,

$$u = c^2 - t^2, \quad v = 2ct$$

$$(-\infty < t < +\infty).$$

These are the equations of the transformed line, represented in the Cartesian coordinates, in a parametric form. If we eliminate the parameter t from these, we get

$$v^2 = 4c^2 (c^2-u),$$

which is the equation of a parabola with its axis along the real axis but directed oppositely, with its focus at the origin of coordinates and its

parameter (the semi-latus rectum) $p = 2c^2$. In exactly the same way we

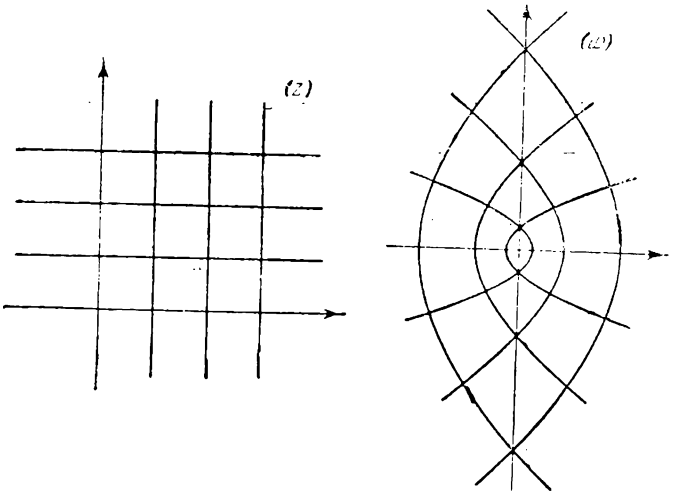


Fig. 11.

establish that every straight line parallel to the real axis $z = t + ic'$ is transformed into the parabola

$$v^2 = 4c'^2 (u + c'^2),$$

with its axis along the positive direction of the real axis, its focus at the origin of coordinates and its parameter $p' = 2c'^2$. Thus, *the two families of straight lines, parallel to the coordinate axes, are mapped by the function* $w = z^2$ *onto two families of parabolas with a common focus at the origin and with their axes along the real axis* (Fig. 11) .From the fact that the families of the straight lines are mutually orthogonal and the mapping is conformal, it follows that the corresponding families of the parabolas are also mutually orthogonal. This can also be easily verified by a direct calculation.

The reader must, of course, bear in mind that the mapping of the whole z-plane by the function $w = z^2$ is not one-to-one. Every point w, different from zero or infinity, has two pre-images. In particular, the pre-images of the parabola $v^2 = 4c^2 (c^2 - u)$ are two straight lines, symmetric with respect to the imaginary axis : $z = c + it$ and $z = -c + it$. Similarly, the pre-images of the parabola $v^2 = 4c'^2 (4 + c'^2)$ are two straight lines, symmetric with respect to the real axis : $z = t + ic'$ and $z = t - ic'$. But, if we take only the image of a half-plane g, bounded by a straight line passing through the origin of coordinates (the interior of an angle with its vertex at the origin of coordinates and its aperture equal to π is such

a half-plane), then, by what has been said above, the correspondence between g and its image d will be one-to-one. The domain d will represent here an angle with its aperture equal to 2π and with its vertex at the origin of coordinates, both the arms of this angle merging into one rectilinear ray, that emanates from the origin of coordinates.

4. GROUP PROPERTIES OF THE BILINEAR TRANSFORMATIONS

In Art. 10 of Chapter 2 it was shown that the bilinear function $w = L(z) = \frac{az+b}{cz+d}$, for which the determinant $ad - bc \neq 0$, maps the extended plane onto itself, conformally with a one-to-one correspondence. We shall now study the properties of this mapping—which itself is called *bilinear*; this is the simplest and the most frequently used type of conformal mapping.

We shall study the set M of all bilinear mappings (or transformations), with their determinants different from zero. Two mappings

$$L_1(z) = \frac{a_1z+b_1}{c_1z+d_1} \quad \text{and} \quad L_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$$

will be regarded identical if, and only if, $L_1(z) = L_2(z)$ for all values of z. For this it will be sufficient if the corresponding coefficients are proportionate, that is, if

$$a_2 = \lambda a_1, \quad b_2 = \lambda b_1, \quad c_2 = \lambda c_1 \text{ and } d_2 = \lambda d_1 \quad (\lambda \neq 0).$$

These conditions are also necessary. In fact, if $L_1(z) = L_2(z)$, we have, in particular,

$$L_1(0) = L_2(0), \quad L_1(1) = L_2(1) \text{ and } L_1(\infty) = L_2(\infty),$$

whence

$$\frac{b_1}{d_1} = \frac{b_2}{d_2} = p, \quad \frac{a_1+b_1}{c_1+d_1} = \frac{a_2+b_2}{c_2+d_2} \text{ and } \frac{a_1}{c_1} = \frac{a_2}{c_2} = q.$$

Putting

$$b_1 = d_1p, \quad b_2 = d_2p \quad a_1 = c_1q \text{ and } a_2 = c_2q$$

into the central equality, we obtain

$$\frac{c_1q+d_1p}{c_1+d_1} = \frac{c_2q+d_2p}{c_2+d_2} \quad \text{or} \quad (c_1d_2-c_2d_1)(q-p) = 0.$$

But $q \neq p$ (otherwise, we would have $\frac{a_1}{c_1} = \frac{b_1}{d_1}$, that is, $a_1d_1 - b_1c_1 = 0$, contrary to our assumption). Hence

$$\frac{c_1}{d_1} = \frac{c_2}{d_2}.$$

The relations obtained above may be written as

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \frac{d_2}{d_1} = \lambda,$$

which was to be proved.

From what has been said above it follows that the value of the determinant of a bilinear mapping is by itself not a characteristic of the mapping in question. In fact, when we change over from the coefficients a_1, b_1, c_1 and d_1 to the coefficients λa_1, λb_1, λc_1 and λd_1 ($\lambda \neq 0$) the determinant gets multiplied by λ^2. Nevertheless, this determinant, being nonzero for any set of values of the coefficients, will remain so even after the transformation.

The mapping

$$U(z) = z,$$

which evidently belongs to the set M, will be called an *identity* or a *unit mapping*.

A mapping *inverse* to the mapping

$$z_1 = L(z) = \frac{az+b}{cz+d}$$

is one under which to every image z_1 there corresponds a pre-image z. The inverse mapping in this particular case is expressed by

$$z = \frac{dz_1 - b}{-cz_1 + a}.$$

We shall denote the mapping inverse to L by the symbol L^{-1}.

If

$$z_1 = L(z) = \frac{az+b}{cz+d} \quad \text{and} \quad z_2 = L_1(z_1) = \frac{a_1z_1+b_1}{c_1z_1+d_1}$$

are any two bilinear mappings (always with nonvanishing determinants), then the mapping obtained by effecting these two in succession, in a definite

order, will be called the *product* of the given mappings. Let us first effect the mapping $z_1 = L(z)$ and then the mapping $z_2 = L_1(z_1)$. Then the product of these two mappings will be denoted by $L_1L(z)$. For this product, we have the expression

$$z_2 = L_1L(z) = \left[a_1\frac{az+b}{cz+d} + b_1\right] : \left[c_1\frac{az+b}{cz+d} + d_1\right] =$$
$$= [(aa_1+cb_1)\,z + (ba_1+db_1)] : [(ac_1+cd_1)\,z + (bc_1+dd_1)].$$

Hence, $z_2 = L_1L(z)$ is also a bilinear mapping; for its determinant, we get

$$(aa_1+cb_1)(bc_1+dd_1) - (ba_1+db_1)(ac_1+cd_1) = (ad-bc)(a_1d_1-b_1c_1) \neq 0.$$

Thus, the mapping $L_1L(z)$ also belongs to the set M.

Obviously

$$LL^{-1}(z) = L^{-1}L(z) = U(z).$$

We note that the mapping $z_2 = LL_1(z)$, obtained by first carrying out the mapping $z_1 = L_1(z)$ and then the mapping $z_2 = L(z_1)$, generally differs from the mapping $L_1L(z)$. Thus, for example, if

$$L(z) = \frac{z}{z+1} \quad \text{and} \quad L_1(z) = \frac{z+1}{z-1},$$

then

$$L_1L(z) = -2z - 1, \quad \text{whereas} \quad LL_1(z) = \frac{z+1}{2z}.$$

The operation of the multiplication of mappings, as definded here, is associative, that is, for arbitrary linear mappings L, L_1 and L_2 we have the relation

$$(LL_1)\,L_2 = L\,(L_1L_2).$$

This property can be readily verified. In fact, let $L_2(z) = z_2$. Then

$$(LL_1)\,L_2(z) = LL_1[L_2(z)] = LL_1(z_2)$$

and

$$L(L_1L_2)(z) = L[L_1L_2(z)] = LL_1(z_2);$$

thus

$$(LL_1)\,L_2(z) = L(L_1L_2)(z).$$

The associative property holds for the product of any number of mappings. This saves us the trouble of writing the superfluous brackets in the product. Thus, for example,

$$L\,[L_1\,(L_2L_3)]\,(z) = LL_1\,(L_2L_3)(z) = L(L_1L_2)\,L_3\,(z) = \ldots = LL_1L_2L_3\,(z).$$

Since the set M contains, besides each pair of mappings L and L_1, their product L_1L (or LL_1) also, and it further contains the inverse mapping L^{-1} along with the mapping L, hence this set constitutes a *group* of mappings (transformations).[1]

5. THE CIRCLE-PRESERVING PROPERTY

We now set out to prove the *circle-preserving property* of the bilinear mapping. This property is expressed in the fact that the image of a straight line or of a circle, under the mapping $w = L(z)$, is again a straight line or a circle ; actually. in this case the image of a straight line may either be a straight line or a circle and similarly that of a circle may either be a straight line or a circle.

This property is self-evident for the linear function $L(z) = \alpha z + \beta$, because the mapping $w = L(z)$ reduces either to a shift (when $\alpha = 1$) or to a rotation and an elongation (if $\alpha \neq 1$) [vide Art. 10, Chapter 2].

We now examine the oft-employed mapping

$$w = \Lambda(z) = \frac{1}{z}.$$

The equation of a straight line or a circle can, of course, be written in the form

$$A(x^2+y^2) + 2Bx + 2Cy + D = 0.$$

We have a straight line when $A = 0$ and B and C are not simultaneously vanishing; we get a circle when $A \neq 0$ and $B^2 + C^2 - AD > 0$. Replacing, in this expression, $x^2 + y^2$ by $z\bar{z}$, $2x$ by $z + \bar{z}$ and $2y$ by $-i(z-\bar{z})$, we express the equation in the form

$$Az\bar{z} + (B-iC)z + (B+iC)\bar{z} + D = 0$$

or

$$Az\bar{z} + \bar{E}z + E\bar{z} + D = 0, \tag{1}$$

where $E = B + Ci$. Here, $A = 0$ and the complex number $E \neq 0$ in the case of a straight line, while $A \neq 0$ and $EE - AD > 0$ in the

[1] See, in this connection, A.G. Kurosh, *loc. cit.*; also Van der Waerden, *Modern Algebra.*

case of a circle. Conversely, any equation of this form with real coefficients A and D and complex conjugate coefficients E and $\bar{E}$ is the equation of a straight line if $A = 0$ and the number E is nonvanishing; it is the equation of a circle if $A \neq 0$ and $E\bar{E} - AD > 0$. To verify this, it is sufficient to change over from z to the variables x and y by the formulae

$$z\bar{z} = x^2 + y^2, \quad z = x + iy \quad \text{and} \quad \bar{z} = x - iy.$$

In order to obtain the image of the curve (1) under the mapping $w = 1/z$, we replace z in the equation (1) by $1/w$. Thus, we obtain

$$A\frac{1}{w\bar{w}} + \bar{E}\frac{1}{w} + E\frac{1}{\bar{w}} + D = 0 \quad \text{or} \quad Dw\bar{w} + Ew + \bar{E}\bar{w} + A = 0. \tag{2}$$

Equation (2) has the same form as equation (1) with A replaced by D, D by A and E by E. It, therefore, follows that for $D = 0$ this is the equation of a straight line (because in this case either $A = 0$ and $E \neq 0$, or $A \neq 0$ and $E\bar{E} - AD = E\bar{E} > 0$, that is, again $E \neq 0$), and for $D \neq 0$ it is the equation of a circle (because, for $A \neq 0$ the equation (1) represents a circle and, consequently, the condition $E\bar{E} - AD > 0$ is satisfied; for $A = 0$ it expresses a straight line and, consequently, E is nonvanishing, whence $E\bar{E} - AD = E\bar{E} > 0$). We have thus shown *that the image of a straight line or a circle under the mapping* $w = \Lambda(z) = 1/z$ *is again either a straight line or a circle.*

Considering any bilinear function

$$w = L(z) = \frac{az+b}{cz+d} \qquad (c \neq 0),$$

we represent it in the form

$$w = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}.$$

We put

$$z_1 = L_1(z) = cz + d, \quad z_2 = \Lambda(z_1) = \frac{1}{z_1} \text{ and } w = L_2(z_2) = \frac{a}{c} + \frac{bc-ad}{c} z_2;$$

then, $L(z)$ is written as the product of three mappings:

$$L = L_2 \Lambda L_1.$$

Since the image of a straight line or a circle under each of the mappings

L_1, Λ and L_2 is a straight line or a circle, so the mapping L would also possess the same property. Thus, the circle-preserving property of a bilinear mapping is fully proved.

If $c \neq 0$, then at the point $\delta = -\frac{d}{c}$ the function $L(z) = \frac{az+b}{cz+d}$ becomes infinity. Hence, the image of any straight line or circle, passing through δ, must contain the point at infinity: $L(\delta) = \infty$, and hence cannot be a circle. By the circle-preserving property this image is then a straight line. The image of a straight line or a circle not passing through the point δ cannot contain the point at infinity and, consequently, cannot be a straight line. By the circle-preserving property this image is here a circle. Thus, *under the mapping $w = L(z)$ all straight lines and circles passing through the point δ are transformed into straight lines in the w-plane, while all straight lines and circles not passing through δ are transformed into circles in the w-plane.*

Let $w = L(z)$ be any bilinear function, γ a straight line or a circle in the z-plane and Γ its image in the w-plane (that is, again a straight line or a circle). We now examine the domains g_1 and g_2 bounded by the line γ in the z-plane. These will either be the two half-planes or else the interior and exterior of a circle. We shall show that the image of one of them will be the first, and that of the other the second, of the two domains bounded by the image line Γ in the w-plane. Let z_1 be a point in the domain g_1 and z_2 a point in the domain g_2. Since z_1 and z_2 do not lie on γ, the images w_1 and w_2 of these points cannot lie on Γ and, hence, fall in the domains into which the w-plane is divided by Γ. If we assume that they fall in one and the same domain, they can be joined by the segment Δ of a straight line (or by an arc of a circle) in this domain, having no points common with Γ (Fig. 12). The pre-image of the segment Δ in the z-plane must be a segment of a straight line (or an arc of a

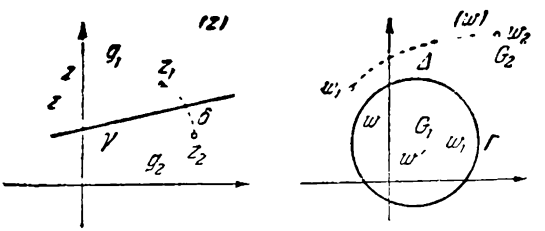


Fig. 12.

circle) δ, connecting z_1 with z_2 and having no points in common with γ. But the existence of such a segment contradicts that z_1 and z_2 lie in the different domains g_1 and g_2. Thus, from the fact that z_1 and z_2 lie in the different domains g_1 and g_2, it follows that their images w_1 and w_2 would

also lie in different domains bounded by the line Γ. We denote by G_1 the domain containing w_1 and by G_2 the domain containing w_2, and will show that the image of the domain g_1 is G_1 and that of the domain g_2 is G_2.

Now, if z', for example, be any point in the domain g_1, then from the statement that z' and z_2 lie in the different domains g_1 and g_2, it follows, by what has been proved above, that their images w' and w_2 lie in the different domains G_1 and G_2. But w_2 belongs to G_2 ; hence, w' belongs to G_1. Thus, the image of every point of the domain g_1 belongs to the domain G_1 and in the same manner the image of every point of the domain g_2 belongs to G_2. Finally, we take any arbitrary point w in the domain G_1. It must be the image of one of the points z in the domain g_1 or g_2. But z cannot be a point of the domain g_2 because otherwise w would be a point of the domain G_2. Hence, w is the image of a point z belonging to g_1. Thus, G_1 is the image of the domain g_1 and G_2 that of g_2. Hence, we have shown that the two domains bounded by the line γ are mapped onto two domains bounded by the line Γ ; besides, in order to recognize as to which of the two domains bounded by the line Γ is the image of the given domain g_1 bounded by the line γ, it is sufficient to track the image w_1 of only one point $z_1 \in g_1$. The domain G_1, to which w_1 belongs, will also be the image of the domain g_1.

6. INVARIANCE OF THE CROSS RATIO

If the bilinear transformation $w = L(z)$ is different from an identity transformation, then w, in general, is different from z. However, in this case there are certain *stationary points of transformation*, characterized by the equation

$$z = L(z) = \frac{az+b}{cz+d}.$$

First, let $c = 0$ $(d \neq 0)$. Then, $L(z)$ represents a linear function :

$$l(z) = \alpha z + \beta \quad (\alpha = a/d, \ \beta = b/d).$$

Since $l(\infty) = \infty$, hence one of the stationary points of the linear transformation is the point at infinity : $z = \infty$. For $\alpha \neq 1$, there exists another stationary point given by the equation

$$z = \alpha z + \beta,$$

namely, the point $z = \frac{\beta}{1-\alpha}$. For $\alpha = 1$ and $\beta \neq 0$, there is no finite

stationary point. But if $\alpha \neq 1$, $\beta \neq 0$ and $\alpha \to 1$, then the finite stationary point $\frac{\beta}{1-\alpha}$ tends to the point at infinity. Hence, in the transformation $l(z) = z + \beta$ $(\beta \neq 0)$ the point at infinity may be looked upon as two stationary points merged together.

Now, let $c \neq 0$. Then $L(\infty) = a/c \neq \infty$, i.e. the point $z = \infty$ is not a stationary point. Exactly in the same way, the point $-d/c$ is also not stationary, because

$$L\left(-\frac{d}{c}\right) = \infty \neq -\frac{d}{c}.$$

For the case $z \neq \infty$ and $z \neq -\frac{d}{c}$, let us solve the equation

$$z = \frac{az+b}{cz+d},$$

that is,

$$cz^2 - (a-d)z - b = 0.$$

We obtain

$$z = \frac{a-d+\sqrt{(a-d)^2+4bc}}{2c}.$$

If $(a-d)^2 + 4bc \neq 0$, we have two different finite stationary points. For $(a-d)^2 + 4bc = 0$, these two points merge into a single finite stationary point: $\frac{a-d}{2c}$.

Thus, *a bilinear transformation, different from the identity transformation, has only two stationary points, which in a special case might merge into a single point.* On the other hand, linear transformations are completely characterized by the fact that at least one of the stationary points is the point at infinity.

A bilinear transformation having more than two stationary points may be only the identity transformation $U(z) = z$ (for which actually all points are stationary). It can thus be derived that for the coincidence of two bilinear transformations $L(z)$ and $\Lambda(z)$, it is sufficient that the equation $L(z) = \Lambda(z)$ be satisfied for three different points z: z_1, z_2 and z_3. Now, let $L(z_k) = \Lambda(z_k) = w_k$ $(k=1, 2, 3)$. Then, $\Lambda^{-1}(w_k) = z_k$ $(k=1, 2, 3)$ and, consequently, the transformation $\Lambda^{-1} L(z)$ takes over the points z_k once again to those very points, i.e. $\Lambda^{-1} L(z_k) = z_k$ $(k=1, 2, 3)$ [since

$L(z_k) = w_k$ and $\Lambda^{-1}(w_k) = z_k$]. Hence, the transformation $\Lambda^{-1} L$ has three different stationary points: z_1, z_2 and z_3, i.e. it is an identity transformation:

$$\Lambda^{-1} L = U.$$

Thus

$$\Lambda(\Lambda^{-1} L) = \Lambda U.$$

But

$$\Lambda(\Lambda^{-1} L) = (\Lambda \Lambda^{-1}) L = UL = L$$

and

$$\Lambda U = \Lambda.$$

Finally, we obtain

$$L = \Lambda,$$

as was to be proved.

Thus, *to define a bilinear transformation, it is sufficient to assign three different points*: w_1, w_2 *and* w_3, *onto which the three given points* z_1, z_2 *and* z_3 *are mapped.*

Now, let us determine a bilinear function effecting such a transformation, and assume at the outset that z_1, z_2 and z_3 are finite points and that $w_1 = 0$, $w_2 = \infty$ and $w_3 = 1$.

In order that the bilinear function

$$w = \Lambda(z) = \frac{az+b}{cz+d}$$

might reduce to zero for $z = z_1$ and become infinity for $z = z_2$, it is necessary and sufficient that in the first case $z = z_1$ be the zero of the numerator $az + b$, i.e. the numerator may have the form $a(z-z_1)$, while in the second case $z = z_2$ be the zero of the denominator, i.e. the denominator may have the form $c(z-z_2)$. Hence, the required function must be of the form

$$w = \frac{a}{c} \cdot \frac{z-z_1}{z-z_2}.$$

But, for $z = z_3$, w must be equal to unity, i.e.

$$1 = \frac{a}{c} \cdot \frac{z_3-z_1}{z_3-z_2},$$

whence we obtain

$$\frac{a}{c} = 1 : \frac{z_3 - z_1}{z_3 - z_2},$$

and, hence,

$$w = \Lambda(z) = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

This is the required bilinear function, transforming the points z_1, z_2 and z_3 into the points 0, ∞ and 1, respectively.

Now, let w_1, w_2 and w_3 be arbitrary (different) finite points and $w = L(z)$ be a bilinear transformation satisfying the conditions

$$L(z_1) = w_1, \quad L(z_2) = w_2 \quad \text{and} \quad L(z_3) = w_3.$$

As already proved, the function $\zeta = \Lambda_1(w) = \frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2}$ transforms the points w_1, w_2 and w_3 into the points 0, ∞ and 1. Hence, the function $\Lambda_1 L(z)$ transforms the points z_1, z_2 and z_3 into the points 0, ∞ and 1, i.e.

$$\Lambda_1 L(z) = \Lambda(z) = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

From the relation

$$\Lambda_1 L = \Lambda,$$

it follows that

$$\Lambda_1^{-1}(\Lambda_1 L) = \Lambda_1^{-1}\Lambda,$$

that is,

$$L = \Lambda_1^{-1}\Lambda$$

[for $\Lambda_1^{-1}(\Lambda_1 L) = (\Lambda_1^{-1}\Lambda_1)L = UL = L$].

The last relation solves the problem, since the transformations Λ and Λ_1 are known to be

$$\Lambda(\zeta) = \frac{\zeta - z_1}{\zeta - z_2} : \frac{z_3 - z_1}{z_3 - z_2} \quad \text{and} \quad \Lambda_1(\zeta) = \frac{\zeta - w_1}{\zeta - w_2} : \frac{w_3 - w_1}{w_3 - w_2}.$$

However, for studying the transformation $w = L(z)$ it is more convenient to use directly the relation

$$\Lambda_1 L(z) = \Lambda(z),$$

whence, on replacing $L(z)$ by w, one gets

$$\Lambda_1(w) = \Lambda(z),$$

or

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}. \tag{3}$$

This gives the bilinear function $w = L(z)$ in an implicit form.

We have solved the problem under consideration on the assumption that all the points z_1, z_2, z_3, w_1, w_2 and w_3 are finite. If, say, $z_1 = \infty$, then the function $\Lambda(z)$, transforming the points $z_1 = \infty$, z_2 and z_3 into points $w_1 = 0$, $w_2 = \infty$ and $w_3 = 1$, takes the form :[1]

$$\Lambda(z) = \frac{1}{z-z_2} : \frac{1}{z_3-z_2}.$$

Hence, equation (3) gets replaced by the equation

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{1}{z-z_2} : \frac{1}{z_3-z_2} \tag{4}$$

(assuming that the points w_1, w_2 and w_3 are finite). If $z_2 = \infty$, then the function $\Lambda(z)$, transforming the points z_1, $z_2 = \infty$ and z_3 into the points $w_1 = 0$, $w_2 = \infty$ and $w_3 = 1$, takes the form

$$\Lambda(z) = (z-z_1) : (z_3-z_1)$$

and, consequently, equation (3) gets replaced by the equation

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = (z-z_1) : (z_3-z_1). \tag{5}$$

If $z_3 = \infty$, then the function $\Lambda(z)$, transforming the points z_1, z_2 and $z_3 = \infty$ into the points $w_1 = 0$, $w_2 = \infty$ and $w_3 = 1$, takes the form

$$\Lambda(z) = \frac{z-z_1}{z-z_2};$$

[1] The expression that follows can be obtained if, considering z_1 to be finite, we rewrite

$$\frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}$$

in the form

$$\frac{\frac{z}{z_1}-1}{z-z_2} : \frac{\frac{z_3}{z_1}-1}{z_3-z_2}$$

and then take the limit as $z_1 \to \infty$.

equation (3) then gets replaced by the equation

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2}. \tag{6}$$

In the same way, the left-hand side of equation (3) should be replaced by

$$\frac{1}{w-w_2} : \frac{1}{w_3-w_2},$$

$$(w-w_1) : (w_3-w_1),$$

or

$$\frac{w-w_1}{w-w_2}$$

depending on whether $w_1 = \infty$, $w_2 = \infty$ or $w_3 = \infty$.

Thus, we arrive at the following mnemonic rule: if $z_k = \infty$, or $w_l = \infty$ ($k=1, 2, 3$; $l=1, 2, 3$), then the differences in which z_k or w_l appear in equation (3) must be replaced by unity. The reader will easily verify the validity of this rule by going over to the limit $z_k \to \infty$ or $w_l \to \infty$ in equation (3).

An important general property of bilinear transformations follows from equation (3). Let a, b, c and d be arbitrary finite different complex numbers. We call ratio

$$\frac{c-a}{c-b} : \frac{d-a}{d-b}$$

the *cross (or an harmonic) ratio of the four numbers or of the four points a, b, c and d.* This ratio will be represented by the symbol (a, b, c, d) :

$$(a, b, c, d) = \frac{c-a}{c-b} : \frac{d-a}{d-b}.$$

We shall extend the definition of the cross ratio also to the case when one of the four points a, b, c and d is at infinity. In fact, by the cross ratio of four points (including the point at infinity as one of them) we shall mean the limit of the cross ratio of four finite points, of which three coincide with the three given points while the fourth one tends to the point at infinity. Following this definition, we get

$$(\infty, b, c, d) = \frac{1}{c-b} : \frac{1}{d-b},$$

$$(a, \infty, c, d) = (c-a) : (d-a),$$

$$(a, b, \infty, d) = 1 : \frac{d-a}{d-b},$$

$$(a, b, c, \infty) = \frac{c-a}{c-b}.$$

Now, let $w = L(z)$ be any bilinear transformation. We denote by A, B, C and D the points into which it transforms the four different points a, b, c and d. Since the three points a, b and d are transformed into the points A, B and D, therefore $w\ [= L(z)]$ and z will be connected by the relation (3) :

$$\frac{w-A}{w-B} : \frac{D-A}{D-B} = \frac{z-a}{z-b} : \frac{d-a}{d-b},$$

where the differences in which the point at infinity appears have to be replaced by unity. When $z = c$, we must put $w = C$ (because in the transformation under consideration the point c corresponds to the point C). Consequently,

$$\frac{C-A}{C-B} : \frac{D-A}{D-B} = \frac{c-a}{c-b} : \frac{d-a}{d-b}$$

(again, the differences in which the point at infinity appears have to be replaced by unity). Hence

$$(A, B, C, D) = (a, b, c, d). \tag{7}$$

Thus, *under a bilinear transformation the cross ratio of any given four points does not change ; in other words, the cross ratio remains invariant under a bilinear transformation.*

7. MAPPING OF DOMAINS BOUNDED BY STRAIGHT LINES OR CIRCLES

With the help of the circle-preserving property of a bilinear transformation and the possibility of mapping any triad of points z_1, z_2 and z_3 onto another, previously given, triad w_1, w_2, w_3, we shall prove the following proposition :

Whatever be the straight lines or circles γ *and* Γ *and the two triads of points* z_1, z_2, z_3 *and* w_1, w_2, w_3 *belonging respectively to* γ *and* Γ, *there exists a bilinear function* $w = L(z)$ *which maps* γ *onto* Γ *such that, the points* z_1, z_2 *and* z_3 *are mapped onto* w_1, w_2, *and* w_3, *respectively.*

Actually, we construct the bilinear function $w = L(z)$ satisfying the conditions $L(z_j) = w_j$ $(j = 1, 2, 3)$. According to the foregoing considerations, there does exist such a function and it is a unique one satisfying these conditions. This function maps the straight line or circle γ onto some straight line or circle Γ'. But as γ passes through the points z_1, z_2 and z_3, Γ' passes through the points w_1, w_2 and w_3 and since we cannot draw two different straight lines or circles through three given points, Γ' must coincide with Γ. Thus, $w = L(z)$ satisfies all the requirements of the above proposition.

We again take arbitrary straight lines or circles γ and Γ (different or coincident) and let g be one of the two domains bounded by the line γ, and G be one of the two domains bounded by Γ. Evidently, both of these may be half-planes, interior, or exterior of a circle. We choose any arbitrary triad of points z_1, z_2 and z_3 on γ and assume, for the sake of definiteness, that when the observer moves along γ in the direction from z_1 to z_3 through the point z_2 the domain g lies to his left. Let w_1, w_2 and w_3 be a triad of points on Γ such that for an observer moving along the curve Γ from the point w_1 to w_3 through the point w_2 the domain G lies to his left. In other respects, the points w_1, w_2 and w_3 are arbitrary. We form, in a manner shown before, a bilinear function $w = L(z)$ satisfying the conditions: $w_j = L(z_j)$ $(j = 1, 2, 3)$ and, consequently, mapping γ onto Γ. We shall show that this function also maps the domain g onto the domain G. In fact, if δ be the segment of the normal to the line γ, drawn through the point z_2 towards the interior of the domain g, i.e. to the left of the observer, at the point z_2 and looking along γ in the direction defined above, then due to the conformality of the mapping $w = L(z)$ the image Δ of this segment (this will be either a segment of a straight line or an arc of a circle) will lie to the left of the observer at the point w_2 and looking along Γ in the direction defined above (see Fig. 13). Hence, by the conditions assumed, Δ will belong to G. Thus,

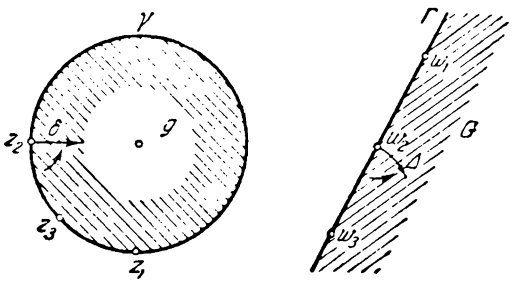


Fig. 13

we have already established that the domain G contains the images of certain points belonging to g (for instance, the images of the points of the segment δ). But, by Art. 5, $L(g)$ is one of the domains whose boundary coincides with the image of the boundary of the domain g, *i.e.* with $\Gamma = L(\gamma)$. Since there are only two such domains and one of them G contains the images of the points of the domain g, this is the required image of the domain g: $G = L(g)$.

We shall now explain what has been said above by an illustration. Suppose we have to map conformally the upper half-plane $y > 0$ onto the interior of a unit circle.

To solve this problem, we assume, for instance, that $z_1 = -1$, $z_2 = 0$ and $z_3 = 1$ so that the half-plane lies to the left of an observer passing along the real axis in the direction from z_1 to z_3 through z_2, and also choose on the unit circle three points w_1, w_2 and w_3 such that the interior of the circle lies to the left of the observer passing along the circumference of the circle in the direction from w_1 to w_3 through w_2. For the sake of simplicity, we can take $w_1 = 1$, $w_2 = i$ and $w_3 = -1$. Then the linear mapping satisfying the conditions $w_j = L(z_j)$ $(j=1, 2, 3)$ will be the actual mapping sought for. This can be represented in the form

$$\frac{w-1}{w-i} : \frac{-1-1}{-1-i} = \frac{z+1}{z} : \frac{1+1}{1},$$

or

$$w = \frac{z-i}{iz-1}.$$

8. SYMMETRY AND ITS PRESERVATION

Let z_1 and z_2 be two points, symmetric about a certain straight line γ. Then any circle δ passing through z_1 and z_2 will have its centre on γ and, consequently, will be orthogonal to γ. The straight line passing through z_1 and z_2 would also be orthogonal to γ. It can be readily seen that the converse is also true, i.e. if any circle or straight line passing through the pair of points z_1 and z_2 is orthogonal to γ, then z_1 and z_2 are symmetric about γ.

We map the plane by a bilinear function $w = L(z)$ such that the straight line γ goes over to a certain straight line or a circle Γ. Then the pair of points z_1 and z_2, symmetric about γ, passes over to a certain pair of points w_1 and w_2 and every circle or straight line δ passing through z_1 and z_2 transforms into a circle or straight line Δ passing through w_1 and w_2; conversely, any straight line or circle passing through w_1 and w_2 will be the image of a certain straight line or circle passing through z_1 and z_2. By virtue of the symmetry of the points z_1 and z_2 relative to γ and the conformality of the mapping $w = L(z)$, straight lines and circles passing through w_1 and w_2 will all be orthogonal to Γ. If, therefore, Γ be a straight line, the points $w_1 = L(z_1)$ and $w_2 = L(z_2)$ will be symmetric with respect to Γ. Generalizing the concept of symmetry, we call two points *symmetric* with respect to a circle if any straight line or circle passing through them is orthogonal to the given circle. Then we can say that if the straight line γ is mapped by a bilinear function $w = L(z)$ onto the circle Γ, then any pair of points symmetric with respect to this straight line is mapped onto a certain pair of points symmetric with respect to the circle, and conversely.

Hence, it follows that a point symmetric to w_1 with respect to Γ is uniquely determined by the circle Γ and the point w_1. In fact, had there existed two different points w_2 and w_2' symmetric to w_1 with respect to Γ, then, under the bilinear mapping of Γ onto the straight line γ, the points w_1, w_2 and w_2' would have passed over to points z_1, z_2 and $z_2' \neq z_2$, while z_1 and z_2 as well as z_1 and z_2' would have been symmetric with respect to γ, which is evidently impossible.

The foregoing logic of mapping the straight line γ onto a straight line or a circle Γ can be carried over without any changes to the case when a circle is mapped onto a circle, and we find that under a bilinear mapping any pair of points symmetric with respect to the circle, γ is mapped onto a pair of points symmetric with respect to the circle Γ which is the image of γ. Thus, we would arrive at the following general *symmetry-preserving* property of the bilinear transformations :

If the points z_1 and z_2 are symmetric with respect to a certain straight line or circle γ, then under any bilinear mapping $w = L(z)$ their images w_1 and w_2 will be symmetric with respect to the image : $\Gamma\ [= L(\gamma)]$ *of* γ.

We note a special case of this proposition. Let γ be mapped onto the circle Γ and let z_1 be a point mapped in this case onto the centre w_1 of the circle Γ. Then the point z_2, symmetric to z_1, must be mapped onto the point w_2 of the extended plane, symmetric to w_1 with respect to Γ. But such a point (w_2) is the point at infinity. In reality, the straight line connecting w_1 and $w_2 = \infty$, i. e. any straight line passing through the centre of the circle Γ, is orthogonal to Γ.

By virtue of the uniqueness of the symmetric point (symmetric to the given point with respect to the given circle), the point at infinity, and only that, will be symmetric to the centre of the circle Γ with respect to Γ.

Let γ be any arbitrary straight line or circle. The mapping of the extended plane, as a result of which any point z is mapped onto a point z^*, symmetric to z with respect to γ, is called a *transformation* of *symmetry* with respect to γ or a *mirror reflection in* γ. When γ is a circle, this mapping is also called an *inversion* with respect to that circle.

We shall now give an analytical expression for the transformation of symmetry. First, let γ be a straight line. This is completely defined by one of its points a and the unit vector $\alpha = \cos\theta + i \sin\theta$ directed in the sense of the straight line.

We carry out the transformation

$$z = a + \alpha w = l(w),$$

which evidently maps the real axis of the w-plane onto the straight line in question. Since the mapping $w = l^{-1}(z)$ transforms γ into the real

axis, so it transforms every pair of points z and z^*, symmetric with respect to γ, into a pair of points w and w^*, symmetric with respect to the real axis. The latter are expressed by the conjugate complex numbers $w = t$ and $w^* = \bar{t}$. Hence, $z - a = \alpha t$ or $z - a = \overline{\alpha t} = \bar{\alpha}\bar{t}$ and $z^* - a = \alpha w^* = \bar{\alpha}\bar{t}$. Eliminating t from the last two equalities, we obtain

$$z^* - a = \alpha^2 \overline{(z-a)}. \tag{8}$$

This equation shows that for carrying out the transformation of symmetry with respect to the straight line γ, passing through the point a and inclined at an angle θ to the real axis, we have to pass over from the vector $z - a$ to the vector $\overline{z-a}$, symmetric to it with respect to the real axis, and then rotate the latter vector about the point a through an angle 2θ.

Next, we study the transformation of symmetry with respect to the circle Γ with its centre at a and radius R $(0<R<\infty)$. We carry out a bilinear transformation, mapping Γ onto the real axis. It would be most convenient to consider the mapping

$$z = a + R\frac{1+iw}{1-iw} = L(w),$$

which puts the points $z_1 = a - iR$, $z_2 = a + R$ and $z_3 = a + iR$ of the circle Γ in correspondence with the points $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$ of the real axis, and thereby maps the real axis onto Γ. The inverse mapping, $w = L^{-1}(z)$, transforms Γ into the real axis and transforms every pair of points z and z^*, symmetric with respect to Γ, into the pair of points w and w^*, symmetric with respect to the real axis. Since w and w^* are expressed by the conjugate complex numbers $w = t$ and $w^* = \bar{t}$, so $z - a = R\frac{1+it}{1-it}$ or $\overline{z-a} = R\frac{1-i\bar{t}}{1+i\bar{t}}$ and $z^* - a = R\frac{1+i\bar{t}}{1-i\bar{t}}$. Multiplying the last two relations termwise, we get

$$\overline{(z-a)}(z^*-a) = R^2$$

or

$$z^* - a = \frac{R^2}{\overline{z-a}}. \tag{9}$$

In particular, the transformation of symmetry with respect to the unit circle $|z| = 1$ has the form $z^* = 1/\bar{z}$. We have already studied this in Art. 4 of Chapter 1.

It follows from (9) that firstly

$$\operatorname{Arg}(z^*-a) = -\operatorname{Arg}(\overline{z-a}) = \operatorname{Arg}(z-a)$$

and secondly

$$|z^* - a| \cdot |z - a| = R^2.$$

Hence, the points z^* and z lie on one and the same ray, issuing from the centre, and at such distances from the centre that their product is equal to the square of the radius. These two conditions, or, what is the same thing, the formula (9), determine the position of one of the two points z and z^* from that of the other, i.e. an inversion with respect to the circle $|z - a| = R$. We leave it to the reader to establish this property of the points, symmetric with respect to a circle, in terms of elementary geometrical considerations.

It follows from equations (8) or (9) that *the general transformation of symmetry reduces to a linear (or bilinear) transformation followed by a transformation of symmetry with respect to the real axis.*

Thus, for instance, the transformation of symmetry with respect to a straight line can be represented in the form

$$z_1 = \bar{a} + \bar{\alpha}^2 (z-a) \tag{10}$$

and

$$z^* = \bar{z}_1,$$

while one with respect to a circle can be written as

$$z_1 = \bar{a} + \frac{R^2}{z-a}$$

and

$$z^* = \bar{z}_1. \tag{11}$$

Since a linear transformation is conformal and possesses the circle-preserving property and the transformation of symmetry with respect to the real axis possesses the same property, with the only difference that while preserving the magnitudes of the angles this transformation inverts the sense of the angles, hence it follows that the transformation of symmetry possesses the above mentioned properties in the most general case, that is, it is a conformal mapping of the second kind and transforms straight lines and circles into straight lines or circles.

9. EXAMPLES

We shall illustrate the property of symmetry preservation under a bilinear mapping by two examples.

Example 1. Suppose we have to map conformally the upper half-plane onto the interior of the circle $|w| < R$, so that a given point α of the half-plane is mapped onto the centre of the circle : $w = 0$.

The unknown function, if it exists, would vanish for $z = \alpha$, i.e. $L(\alpha) = 0$. Thus, we know the zero, $z = \alpha$, of the function $L(z)$. But the point $\bar{\alpha}$, symmetric to α with respect to the real axis, must be mapped onto a point symmetric to the centre of the circle with respect to the circle itself, i.e. the point at infinity. Hence, $L(z)$ has the form

$$w = L(z) = \frac{a(z-\alpha)}{c(z-\bar{\alpha})} = \lambda \cdot \frac{z-\alpha}{z-\bar{\alpha}}, \tag{12}$$

where λ is a complex number, different from zero.

We shall now show that the function obtained here maps the half-plane onto the circle $|w| < |\lambda|$, so that the point α goes over to the centre of the circle : $w = 0$. This last condition is evidently satisfied by the function (12) for any λ. It remains to verify that under this mapping the real axis passes over to a circle of radius $|\lambda|$, with its centre at the origin of coordinates. In fact, if $z = x$ is an arbitrary real number, the numbers $x - \alpha$ and $x - \bar{\alpha}$ are complex conjugate and, hence,

$$|w| = |L(x)| = \left|\lambda \frac{x-\alpha}{x-\bar{\alpha}}\right| = |\lambda| \left|\frac{x-\alpha}{\overline{x-\alpha}}\right| = |\lambda|.$$

We have thus found that the images of all the points of the real axis lie on the circle $|w| = |\lambda|$ and from this, by virtue of the circle-preserving property, it follows that the image of the real axis is this circle.

In order to obtain the mapping of the half-plane onto a circle of radius R, we take $|\lambda| = R$. The argument of the number λ still remains undetermined. The geometrical meaning of this indeterminateness is fully clear. A change from one value of λ in formula (12) to another with its modulus $|\lambda| = R$ unchanged is equivalent to a change in the arguments of all the points by the same magnitude, i.e. to a rotation of the circle about its own centre $w = 0$. Under such a rotation, the circle passes over onto itself, its centre remains fixed and the conditions of the problem are not at all affected.

If we wish the problem set here to have a unique solution, we must introduce an additional condition. We might stipulate, for instance, that

(*a*) the given point $x = x_0$ of the real axis goes over to the point $w = R$ of the circle, or

(*b*) the derivative $L'(\alpha)$ is a real positive number ; geometrically, it means that the tangents to the curve passing through the point α must not change their slope as a result of this mapping.

Now, under the condition (*a*) we obtain from (12)

$$R = L(x_0) = \lambda \frac{x_0 - \alpha}{x_0 - \bar{\alpha}};$$

hence,

$$\lambda = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha}$$

and

$$L(z) = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \cdot \frac{z - \alpha}{z - \bar{\alpha}}. \tag{13}$$

Obviously,

$$|\lambda| = R \left| \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \right| = R.$$

Now, let $\alpha = \xi + i\eta$, $\eta > 0$. Since

$$L'(z) = \frac{\lambda}{\alpha - \bar{\alpha}},$$

therefore we find, for the condition (*b*), that $\frac{\lambda}{\alpha - \bar{\alpha}} = \frac{\lambda}{2\eta i}$ and, hence, $\frac{\lambda}{i}$ is also a positive real number. But, on the other land, the modulus $|\lambda|$ must be equal to R. Hence, $\lambda = iR$ and

$$L(z) = iR \frac{z - \alpha}{z - \bar{\alpha}}. \tag{14}$$

Example 2. Let us map the circle $|z| < R$ conformally onto itself, so that the given point $z = \alpha$ of the circle passes over to the centre of the circle : $w = 0$.

The required bilinear function $L(z)$, if it exists, vanishes for $z = \alpha$, i.e. $L(\alpha) = 0$. Thus, we see that $z = \alpha$ is the zero of the function $L(z)$. But the point α^*, symmetric to α with respect to the circle $|z| = R$, must go over to a point symmetric to the centre of the circle with respect to the same circle, i.e. to the point at infinity. Hence, the bilinear function $L(z)$

must have the form

$$w = L(z) = \lambda \frac{z-\alpha}{z-\alpha^*},$$

where λ is a complex number, different from zero. By formula (9), the point α^*, symmetric to the point α with respect to the circle $|z| = R$, is

$$\alpha^* = \frac{R^2}{\bar{\alpha}}.$$

Hence

$$w = L(z) = -\lambda\bar{\alpha}\frac{z-\alpha}{R^2-\bar{\alpha}z} = \mu\frac{z-\alpha}{R^2-\bar{\alpha}z}. \tag{15}$$

We shall now show that the function obtained here maps the circle $|z| < R$ onto the circle $|w| < |\mu|/R$, so that the point α goes over to the centre O of the circle. This, evidently, holds for the function $w = L(z)$, for any μ. It now remains to show that the circle $|z| = R$ is mapped by (15) onto the circle $|w| = |\mu|/R$. Let

$$z = R\zeta,$$

where

$$\zeta = \cos\theta + i\sin\theta \qquad (0<\theta<2\pi)$$

be any point of the circle $|z| = R$. Then, for its image

$$w = L(R\zeta) = \mu\frac{R\zeta-\alpha}{R^2-\bar{\alpha}R\zeta} = \frac{\mu}{R\zeta}\frac{R\zeta-\alpha}{R\bar{\zeta}-\bar{\alpha}},$$

we have

$$|w| = \left|\frac{\mu}{R\zeta}\right|\left|\frac{R\zeta-\alpha}{R\bar{\zeta}-\bar{\alpha}}\right| = \frac{|\mu|}{R}$$

(because $|\zeta| = 1$ and $\left|\frac{R\zeta-\alpha}{R\bar{\zeta}-\bar{\alpha}}\right| = 1$). It then follows that the circle $|w| = |\mu|/R$ is the image of the circle $|z| = R$.

In order to obtain the mapping of a circle of radius R onto itself, we have to take in formula (15) $|\mu| = R^2$. The argument of the number μ still remains indeterminate. In order that this problem of mapping might have a unique solution we must stipulate one of the following additional conditions :

(*a*) that the given point a of the circle $|z| = R$ passes over to the point $w = R$ of the same circle, and

(*b*) that the derivative $L'(\alpha)$ is a real positive number.

We leave it to the reader to verify that under the condition (*a*)

$$L(z) = R\frac{R^2-\bar{\alpha}a}{a-\alpha}\cdot\frac{z-\alpha}{R^2-\bar{\alpha}z}, \tag{16}$$

and under the condition (*b*)

$$L(z) = \frac{R^2(z-\alpha)}{R^2-\bar{\alpha}z}. \tag{17}$$

10. THE JOUKOWSKI FUNCTION

Let us examine the function $w = \frac{1}{2}\left(z+\frac{1}{z}\right) = \lambda(z)$. As this function was used by N. E. Joukowski in aerodynamics, it is referred to as the *Joukowski function*. Obviously, the equation $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$ does not have more than two roots z_1 and z_2, for any w. Since $\lambda(z) = \lambda\left(\frac{1}{z}\right) = w$, so $z_1z_2 = 1$. If one of the roots belongs to the interior of the unit circle the other will belong to its exterior, and conversely. Hence, the sets of values of $w = \lambda(z)$ ascribable to the interior or the exterior of the unit circle must be the same.

In order to investigate the mapping $w = \lambda(z)$ in detail, we shall determine the images of the circles $|z| = r$ and of the radii $\operatorname{Arg} z = \alpha + 2k\pi$ (Fig. 14). In this case, we shall confine ourselves, say, to the interior of the (unit) circle $|z| < 1$.

Let us put

$$z = r(\cos t + i\sin t) \quad (0\leqslant t\leqslant 2\pi, \quad 0<r<1);$$

then

$$w = \frac{1}{2}(z+z^{-1}) = \frac{1}{2}\left(\frac{1}{r}+r\right)\cos t - i\frac{1}{2}\left(\frac{1}{r}-r\right)\sin t$$

or

$$u = \frac{1}{2}\left(\frac{1}{r}+r\right)\cos t, \quad v = -\frac{1}{2}\left(\frac{1}{r}-r\right)\sin t \quad (0\leqslant t\leqslant 2\pi). \tag{18}$$

Eliminating the parameter t from the two equations (18), we obtain

$$\frac{u^2}{\left[\frac{1}{2}\left(\frac{1}{r}+r\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(\frac{1}{r}-r\right)\right]^2} = 1. \tag{19}$$

This is the equation of an ellipse with semiaxes $a = \frac{1}{2}\left(\frac{1}{r}+r\right)$ and $b = \frac{1}{2}\left(\frac{1}{r}-r\right)$, and with foci at ± 1. It follows from the formula (18) that

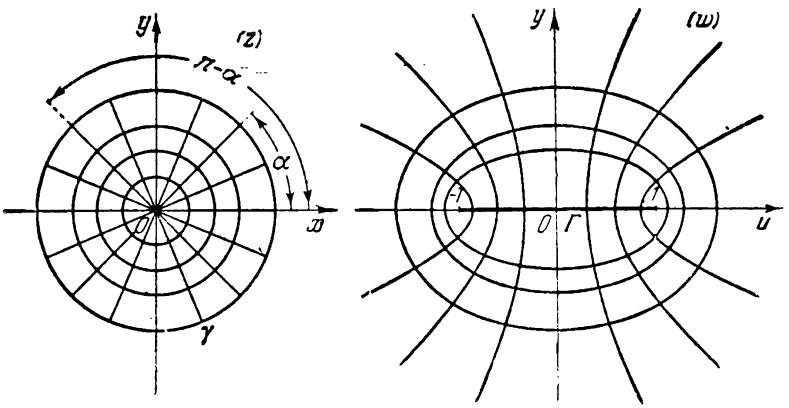


Fig. 14.

when t increases from zero to 2π (i.e. the point z describes once the whole circle $|z| = r$ in the positive direction), the corresponding point (u, v) describes the full ellipse (19) once in the negative direction. In fact, for $0 \leqslant t \leqslant \pi/2$, u is positive and decreases from a to 0, while v is negative and decreases from 0 to $-b$. For $\pi/2 < t < \pi$, u continues to decrease from 0 to $-a$ while v increases from $-b$ to zero; for $\pi < t < 3\pi/2$, u increases from $-a$ to 0 while v increases from 0 to b and, finally, for $3\pi/2 < t < 2\pi$, u increases from 0 to a while v decreases from b to 0.

By varying the radius r of the circle $|z| = r$ from zero to 1, we make a decrease from ∞ to 1 and b from ∞ to 0. The corresponding ellipses run through the totality of ellipses of the w-plane with foci at ± 1. Hence, it follows that $w = \lambda(z)$ maps the unit circle one-to-one onto the domain G which represents the exterior of the segment Γ of the real axis: $-1 \leqslant x \leqslant 1$. In this case, the image of the centre of the unit circle is the point at infinity while the segment Γ described twice is the image of the unit circle.

For the image of the radius $z = t(\cos\alpha + i\sin\alpha)$, $0 \leqslant t < 1$, we obtain, in the first instance, the equation

$$w = \frac{1}{2}\left(\frac{1}{t}+t\right)\cos\alpha - i\frac{1}{2}\left(\frac{1}{t}-t\right)\sin\alpha,$$

that is,

$$u = \frac{1}{2}\left(\frac{1}{t}+t\right)\cos\alpha, \quad v = -\frac{1}{2}\left(\frac{1}{t}-t\right)\sin\alpha \quad (0 \leqslant t < 1). \quad (20)$$

It is obvious from this that the images of two radii, symmetric with respect to the real axis (if one of them corresponds to the angle α, the other does to the angle $-\alpha$), are also symmetric with respect to the real axis, while the images of two radii symmetric with respect to the imaginary axis (if one of them corresponds to the angle α, the other does to the angle $\pi-\alpha$) are also symmetric with respect to the imaginary axis. Hence, it is sufficient to examine only the images of the radii lying in the first quadrant: $0 \leqslant \alpha \leqslant \frac{\pi}{2}$.

We note that when $\alpha = 0$,

$$u = \frac{1}{2}\left(\frac{1}{t}+t\right), \quad v = 0 \quad (0 \leqslant t < 1).$$

This is the complete positive half of the real axis: $1 < u \leqslant \infty$. The interval $-\infty \leqslant u < -1$, symmetric to it, is the image of the radius corresponding to $\alpha = \pi$.

When $\alpha = \frac{\pi}{2}$, we have

$$u = 0, \quad v = -\frac{1}{2}\left(\frac{1}{t}-t\right), \quad (0 \leqslant t < 1).$$

This is the complete negative half of the imaginary axis: $-\infty \leqslant v < 0$. The interval $0 < v \leqslant \infty$ is the image of the radius corresponding to $\alpha = -\frac{\pi}{2}$.

Thus, the image of the 'horizontal' diameter of the unit circle is the infinite interval of the real axis running from the point -1 to the point $+1$ through ∞, and image of the 'vertical' diameter is the entire imaginary axis with the origin of coordinates excluded and the point at infinity included.

Now, let $0 < \alpha < \frac{\pi}{2}$. Then, eliminating the parameter t from equations (20), we get

$$\frac{u^2}{\cos^2 \alpha} - \frac{v^2}{\sin^2 \alpha} = 1. \tag{21}$$

This is the equation of a hyperbola with the real axis $a = \cos \alpha$, the imaginary axis $b = \sin \alpha$ and foci at ± 1. The point w, however, does not describe this hyperbola completely as the point z describes the entire radius $z = t(\cos \alpha + i \sin \alpha)$ $(0 \leqslant t < 1)$. Now, it follows from equations (20) that when t increases from zero to unity u decreases from ∞ to $\cos \alpha$ and v increases from $-\infty$ to zero. Hence, the point describes

but once a quarter of the entire hyperbola lying in the fourth quadrant. By what has been said above the quarter that lies in the first quadrant, i.e. the one symmetric to the foregoing with respect to the real axis, will be the image of the radius symmetric to the given radius with respect to the real axis, i.e. the image of the radius corresponding to the angle $-\alpha$. It would be incorrect to say that any branch of the hyperbola lying in the first and the fourth quadrants is the image of the pair of given radii; as a matter of fact, the vertex of the hyperbola, $u = \alpha$, $v = 0$, does not belong to this image (we should not forget that the radii in question are taken with their end points excluded and the vertex of the hyperbola is the image of each of the points $t = 1$).

We further find that the quarters of the same hyperbola lying in the third and the second quadrants will be the images of the radii corresponding to the angles $\pi - \alpha$ and $\alpha + \pi$ (or $\alpha - \pi$).

The complete hyperbola, with its two vertices excluded, is the image of four radii: $\pm \alpha$, $\pi \pm \alpha$. We note that the image of each of the two diameters composed of these radii will be that part of the hyperbola which is composed of a pair of its quarters symmetric to each other at the point at infinity.

Thus, *the function* $w = \lambda(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ *maps one-to-one the interior as well as the exterior of the unit circle onto the exterior of the segment* $-1 \leqslant u \leqslant +1$ (*of the real axis*).

Here, the circles $|z| = r$ *are mapped onto ellipses with foci at* ± 1 *and semiaxes* $\frac{1}{2}\left|\frac{1}{r} \pm r\right|$, *while the pair of diameters symmetric with respect to the coordinate axes* [*composed of the radii* $z = \pm r(\cos\alpha \pm i \sin\alpha)$ $[0 \leqslant r < 1)$] *are mapped onto hyperbolas with foci at* ± 1 *and semiaxes* $|\cos\alpha|$ *and* $|\sin\alpha|$, *with their vertices excluded.*

Since the derivative of our function

$$w = \lambda'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

does not vanish if $z \neq \pm 1$, the mapping is conformal at all points of the domains examined (i.e. the interior and the exterior of the unit circle). It thus follows that the hyperbolas intersect the ellipses at the same angles at which the radii intersect the circles, i.e. at right angles.

We further examine the images of the circles passing through the points ∓ 1. From

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \lambda(z),$$

we get

$$w-1=\frac{z^2-2z+1}{2z}=\frac{(z-1)^2}{2z} \text{ and } w+1=\frac{z^2+2z+1}{2z}=\frac{(z+1)^2}{2z},$$

whence

$$\frac{w-1}{w+1}=\left(\frac{z-1}{z+1}\right)^2.$$

It is easily seen that this equation is equivalent to the given one. Putting $\frac{z-1}{z+1}=z'$ and $\frac{w-1}{w+1}=w'$, we find that the mapping $w=\lambda(z)$ can be replaced by

$$z'=\frac{z-1}{z+1}, \quad w'=z'^2 \quad \text{and} \quad \frac{w-1}{w+1}=w'. \tag{22}$$

The first of these relations transforms the circles passing through the points ± 1 into straight lines passing through the origin of coordinates, the second transforms these straight lines into rays emanating from the origin of coordinates and finally the last one maps each of these rays onto an arc of a circle, joining the points ± 1. From formula (22) we readily find that if the angle between the circle and the positive direction of the real axis at the point $z=1$ be θ, then the angle, at the point $w=1$, between its image (the arc of a circle) and the positive direction of the real axis will be equal to 2θ (see Fig. 15).

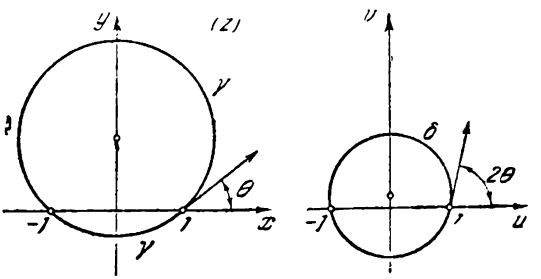


Fig. 15.

Thus, the function $w=\lambda(z)$ maps every circle γ passing through the points ± 1 and making an angle θ, at the point 1, with the positive direction of the real axis onto the arc δ of a circle passing through the points ± 1 and making an angle 2θ with the positive direction of the real axis. The same formulae (22) show that in this case each of the two arcs γ with their ends at ± 1 is separately mapped onto one and the same arc δ.

We note that the exterior of the circle γ under the first of the mappings (22) is transformed into a half-plane, under the second of these

mappings we obtain a domain bounded by rectilinear rays emanating from the origin of coordinates and finally, by the third mapping we get a domain with the arc δ as its boundary. Since all these mappings establish a one-to-one correspondence, the function $w = \lambda(z)$ gives a one-to-one conformal mapping of the exterior of the circle γ (and also of the interior of this circle) onto the domain with the circular arc δ, connecting the points ± 1, as its boundary.

It is useful to note that the function $w = \lambda(z)$ maps the semicircle $k_1 : |z| < 1$, in the upper half-plane onto the lower half-plane w and the semicircle k_2 in the lower half-plane onto the upper half-plane w. But, by the relation $\lambda(z) = \lambda\left(\frac{1}{z}\right)$, the function takes up at the points of the semicircle k_2 the same values that it takes up at the points of the upper half plane that lie in the exterior of the semicircle k_1. If we denote the set of the latter points by K_2 (Fig. 16) we can prove that the upper half-plane is the image of the domain K_2. We finally find that the image of

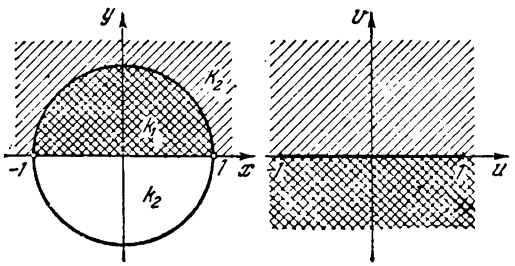


Fig. 16.

the semicircle separating k_1 and K_2 is the segment $-1 < u \leqslant 1$, described only once. Hence, it follows that the image of the upper half-plane under the mapping $w = \lambda(z)$ consists of the upper and the lower half-planes and the segment $-1 < u < 1$ of the real axis, i.e. it happens to be the whole of the plane, excluding the infinite segment of the real axis connecting the points -1 and $+1$ through the point at infinity.

11. DEFINITION OF AN EXPONENTIAL FUNCTION

The exponential function e^x of a real variable x can be defined as the solution of the differential equation $\frac{df}{dx} = f$, satisfying the initial condition $f(0) = 1$. Let us determine an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfying similar conditions :

$$\frac{df}{dz} = f, \text{ with } f(0) = u(0, 0) + iv(0, 0) = 1.$$

Since $\frac{df}{dz} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, the differential equation in question

assumes the form

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = u + iv,$$

whence we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = u, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = v.$$

From the equation $\frac{\partial u}{\partial x} = u$ and the initial condition $u(0, 0) = 1$ we get : $u(x, y) = e^x \lambda(y)$, where $\lambda(y)$ is a differentiable function satisfying the condition $\lambda(0) = 1$. Similarly, from the equation $\frac{\partial v}{\partial x} = v$ and the initial condition $v(0, 0) = 0$, we get $v(x, y) = e^x \mu(y)$, where $\mu(y)$ is a differentiable function satisfying the condition $\mu(0) = 0$.
Hence

$$\frac{\partial v}{\partial y} = e^x \mu'(y) = u = e^x \lambda(y) \quad \text{and} \quad -\frac{\partial u}{\partial y} = -e^x \lambda'(y) = v = e^x \mu(y),$$

that is,

$$\mu'(y) = \lambda(y) \quad \text{and} \quad -\lambda'(y) = \mu(y).$$

Evidently, both $\lambda(y)$ and $\mu(y)$ satisfy one and the same differential equation of the second order, viz.

$$\frac{d^2\varphi(y)}{dy^2} + \varphi(y) = 0,$$

the general solution of which is of the form $\varphi(y) = C_1 \cos y + C_2 \sin y$.

Considering the initial condition for $\mu(y)$, we get $\mu(y) = C \sin y$ and, consequently, $\lambda(y) = \mu'(y) = C \cos y$, whence we get from the initial condition for $\lambda(y)$, $C = 1$,

Thus, $\lambda(y) = \cos y$, $\mu(y) = \sin y$ and consequently

$$f(z) = u(x, y) + iv(x, y) = e^z (\cos y + i \sin y).$$

We have thus arrived at a unique analytic function (this function is analytic in the whole plane, i.e. it is an entire function) satisfying the conditions imposed. This function is called the *exponential function of the complex variable* z and is denoted by $\exp z$:

$$\exp z = e^z (\cos y + i \sin y).$$

We note that for real z ($y=0$, $z=x$)

$$\exp x = e^x,$$

i.e. on the real axis the exponential function of a complex variable coincides with the exponential function of a real variable, which is well known in general analysis.

It can also be directly verified that the *law of multiplication* of exponential functions of real variables ($e^{x_1}e^{x_2}=e^{x_1+x_2}$), can also be extended to exponential functions of complex variables, viz. $\exp z_1 \cdot \exp z_2 = \exp (z_1+z_2)$.

We shall call the complex number z the *exponent* of the function $\exp z$. The foregoing law is then formulated in the following way: *On multiplying two exponential functions, the exponents get added.* These considerations make it possible to use the symbol e^z, concurrently with the symbol $\exp z$. Thus

$$e^z = e^x (\cos y+i \sin y).$$

12. MAPPING BY AN EXPONENTIAL FUNCTION

From the definition of the exponential function,

$$\exp z = e^x (\cos y+i \sin y), \tag{23}$$

it follows that it does not vanish at any point z whatsoever and that

$$|\exp z| = e^x \quad \text{and} \quad \operatorname{Arg} (\exp z) = y + 2k\pi.$$

For $z = iy$ ($x=0$), we get

$$\exp (iy) = \cos y + i \sin y.$$

This relation permits us to use the exponential form

$$c = r \exp (i\varphi) = re^{i\varphi},$$

which is obviously more compact than the trigonometrical form

$$c = r (\cos \varphi+i \sin \varphi) \quad (r\neq 0),$$

for writing a complex number.

We find from the formula (23) that an exponential function has a period $2\pi i$ (because when y is increased by 2π, z is increased by $2\pi i$ and the original value of the function is recovered):

$$\exp (z+2\pi i) = \exp z.$$

We shall show that $2\pi i$ is the *fundamental period* of the function; in other words, any period of the function must be of the form $2k\pi i$ where k is an integer.

Let $\omega = \alpha + \beta i$ be the period of the exponential function. Then

$$\exp(z+\omega) = \exp z$$

for any z and, in particular, for $z = 0$:

$$\exp \omega = \exp(\alpha+i\beta) = e^{\alpha}(\cos\beta + i \sin\beta) = 1.$$

But this means that $e^{\alpha} = 1$, i.e. $\alpha = 0$ and $\cos\beta + i \sin\beta = 1$, i.e. $\beta = 2k\pi$. Hence,

$$\omega = \alpha + i\beta = 2k\pi i,$$

as was to be proved.

We shall regard the expression $\exp \infty$ as meaningless because there does not exist the limit $\lim_{z\to\infty} e^z$. It is sufficient to note that when $x > 0$ and tends to ∞, $e^x \to \infty$, and that when $x < 0$ and tends to $-\infty$, $e^x \to 0$.

From this it follows, in particular, that $\exp z$ does not behave like a polynomial (because any polynomial that is not equal to a constant tends to ∞ as $z \to \infty$). Entire functions different from polynomials, are called *transcendental (entire) functions*; hence, $\exp z$ is a transcendental (entire) function.

Since, by the definition of the function $\exp z$,

$$(\exp z)' = \exp z,$$

the derivative of an exponential function does not vanish for any value of z.

We shall now discuss the geometrical behaviour of the function $w = \exp z$ or, what is the same thing, the mapping effected by it. We have already noted that this function does not take the value $w = 0$ for any value, whatsoever, of z. This means that under the mapping $w = \exp z$ the origin of coordinates of the w-plane does not belong to the image of the finite z-plane. We shall show that any other finite point of the w-plane does belong to this image. In fact, from the equation $w = \exp z$, where $w \neq 0$ is given but $z = x + iy$ is unknown, we obtain $|w| = e^x$, whence we have

$$x = \ln|w| \quad \text{and} \quad \operatorname{Arg} w = y + 2\pi k, \text{ i.e. } y = \operatorname{Arg} w.$$

Thus, the pre-images of the points w can be only those points which are of the form

$$z = \ln|w| + i \operatorname{Arg} w.$$

Evidently, such points are infinitely many because Arg w has an infinite set of values, every pair differing by an integral multiple of 2π. Moreover, each of these points is in fact a pre-image of the point w, because

$$\exp(\ln|w| + i \operatorname{Arg} w) = e^{\ln|w|}(\cos \operatorname{Arg} w + i \sin \operatorname{Arg} w) = \\ = |w|(\cos \operatorname{Arg} w + i \sin \operatorname{Arg} w) = w.$$

Thus, *the set of all the roots of the equation* $w = e^z$ $(w \neq 0)$ *is represented by the formula*

$$z = \ln|w| + i \operatorname{Arg} w = \ln|w| + i(\arg w + 2k\pi), \qquad (24)$$

where $k = 0, \pm 1, \pm 2, \ldots$.

All these points are located on a single straight line, parallel to the imaginary axis, the distance between the neighbouring points being 2π.

We have shown that the function $w = \exp z$ maps the finite z-plane onto the domain obtained from the finite w-plane by excluding therefrom the single point $w = 0$, the mapping here being no longer one-to-one because every point $w \neq 0$ has an infinite set of pre-images, as given by (24).

Since the derivative of the exponential function is everywhere nonzero this mapping is conformal at all points of the finite z-plane.

Let z describe a straight line parallel to one of the coordinate axes (Fig. 17). If this straight line be given by $z = c + it$, i.e. parallel to

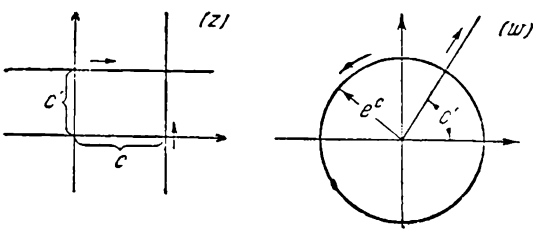


Fig. 17.

the imaginary axis, then $w = e^c(\cos t + i \sin t)$; hence, w will be found on the circle, with centre at the origin of coordinates and radius equal to e^c. In this case, as the point z describes the straight line once (such that the ordinate of this point, which is equal to t, increases continuously from $-\infty$ to $+\infty$), w describes the corresponding circle an infinite number of times in one and the same (positive) direction.

If, however, the point z describes the straight line $z = t + ic'$, parallel to the real axis, then $w = e^t(\cos c' + i \sin c')$, evidently, describes a rectilinear ray emanating from the origin of coordinates and making an angle c' with the positive direction of the real axis. In this case, as

the point z describes the straight line once (such that the abscissa of this point, which is equal to t, increases continuously from $-\infty$ to $+\infty$), w also describes the corresponding ray once in such a way that the distance of this point from the origin of coordinates increases continuously from 0 to ∞ (both these points being, of course, excluded because $|w| = e^t$).

Thus, *under the mapping of the z-plane by the function* $w = e^z$ *the family of straight lines, parallel to the imaginary axis, is mapped onto a family of circles with their centre at the origin of coordinates, and the family of straight lines parallel to the real axis onto a family of rectilinear rays emanating from the origin of coordinates.*

We now examine the domain g, representing the interior of a rectilinear strip of width h ($0 < h \leqslant 2\pi$), parallel to the real axis. Let this strip be bounded by the straight lines $y = \varphi_0$ and $y = \varphi_1$ ($\varphi_1 - \varphi_0 = h$). From

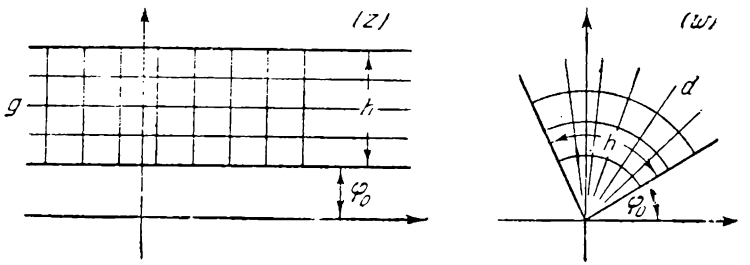


Fig. 18.

what has been established above it follows that the image of the domain g in the w-plane will be the domain d, representing an angle of aperture h, with its vertex at the origin of coordinates and bounded by the rectilinear rays $\operatorname{Arg} w = \varphi_0 + 2k\pi$ and $\operatorname{Arg} w = \varphi_1 + 2k\pi$ (Fig. 18). In this case the correspondence between the points of the domains g and d, established by the function $w = \exp z$, will be one-to-one.

In order to verify the last statement, it is sufficient to note that the pre-images of a certain point w in the domain d will be only the points $\ln|w| + i \operatorname{Arg} w$, which differ from one another only in the values of their imaginary parts. Two such points lie on a single straight line parallel to the imaginary axis and separated by a distance which is an integral multiple of 2π. But our strip g has a width not exceeding 2π and so it can contain in its interior only one pre-image of the point w. Thus, every point $z \in g$ has only one image and every point $w \in d$ only one pre-image in the interior of g : this implies a one-to-one correspondence in the mapping.

We find that *the exponential function* $w = \exp z$ *maps conformally and in a one-to-one manner a strip with width* $h \leqslant 2\pi$, *parallel to the real axis, onto an angle of aperture h with its vertex at the origin of coordinates.*

Hence, whenever we have to map conformally a certain rectilinear strip onto the interior of an angle we have recourse to an exponential function.

If a straight line in the z-plane is not parallel to any of the coordinate axes, its image in the w-plane will neither be a straight line nor a circle, but rather a logarithmic spiral. In fact, if this straight line be

$$z = t(1+i\alpha) + bi, \quad (-\infty<t<+\infty),$$

where α is the slope of the straight line and b the intercept on the imaginary axis, i. e. the ordinate at the origin ($t=0$), then its image will be the curve

$$w = \exp[t+i(\alpha t+b)] = e^t[\cos(\alpha t+b) + i\sin(\alpha t+b)].$$

Here

$$|w| = r = e^t \text{ and } \varphi = \operatorname{Arg} w = \alpha t + b + 2m\pi$$

or, eliminating the parameter t, $r = \exp\frac{\varphi-b-2m\pi}{\alpha}$. But, Arg w or the polar angle φ is defined only to the extent of an integral multiple of 2π. Hence, denoting $\varphi - 2m\pi$ again by φ we obtain

$$r = ce^{\frac{\varphi}{\alpha}}, \quad \text{where } c = e^{-\frac{b}{\alpha}}.$$

This is the equation of a *logarithmic spiral* (Fig. 19). As this spiral is the image of the straight line $z = t(1+i\alpha) + bi$, intersecting the straight

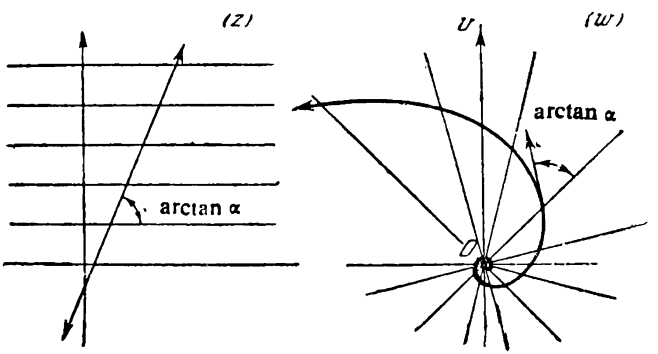


Fig. 19.

lines parallel to the real axis at a fixed angle arctan α, it follows from the conformality of the mapping that the logarithmic spiral itself cuts *at the same angle* the images of the above mentioned straight lines, viz. all the rays emanating from the origin of coordinates. We have thus

found a characteristic property of the logarithmic spiral.

The mappings effected by the functions $w = (z-a)^n$ and $w = \exp z$ exhibit certain similarities between them, which can be disclosed by the formula

$$\exp z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n;$$

we leave it to the reader to prove this formula as an exercise.[1]

We now study the mapping $w = \left(1+\frac{z}{n}\right)^n = \frac{1}{n^n}[z-(-n)]^n$, as a limit of which we obtain the mapping $w = \exp z$.

By what has been said in Art. 3 this function maps an angle of aperture $\frac{h}{n}$ $(0<h\leqslant 2\pi)$ with its vertex at the point $A_n(-n, 0)$, bounded by a part of the real axis $x \geqslant -n$ $(y=0)$ and the ray $\operatorname{Arg}(z+n) = \frac{h}{n} + 2k\pi$, onto an angle of aperture h with vertex at the origin of coordinates and bounded by the rays $\operatorname{Arg} w = 0$ and $\operatorname{Arg} w = h + 2m\pi$ (Fig. 20).

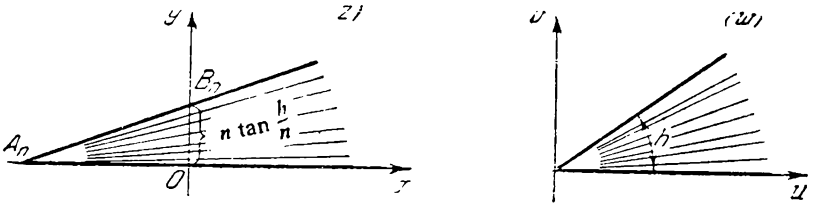


Fig. 20.

As n tends to infinity, the vertex A_n recedes to infinity along the negative part of the real axis and the length of the segment OB_n approaches the limit $\lim_{n\to\infty} n\tan\frac{h}{n} = h$, such that the limiting position of the ray A_nB_n is the straight line $y = h$, binding, along with the real axes, a strip of width h. Here, it is evident that the limiting positions of the rays emanating from the vertex of the angle will be straight lines parallel to the real axis, while the limiting positions of the arcs of the circles with centres at the points A_n will be segments of straight lines perpendicular to the real axis and included within the strip. We shall later see that the mapping by an exponential function may be obtained from the corresponding mapping by a power function by going over to a proper limit.

13. TRIGONOMETRIC FUNCTIONS

We now pass on to the definition of the sine and cosine functions of a complex argument. From the formulas

$$\exp(ix) = \cos x + i\sin x \quad \text{and} \quad \exp(-ix) = \cos x - i\sin x,$$

[1] Vide A. I. Markushevich, *Elements of the Theory of Analytic Functions*, State Teachers Training Publishing House, Moscow, 1944, pp. 106-107.

we obtain the well known Euler's formulae

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2} \quad \text{and} \quad \sin x = \frac{\exp(ix) - \exp(-ix)}{2i},$$

valid for any real value of x. Since the right-hand sides of these formulae are defined for any complex z ($z \neq \infty$) and are, obviously, analytic functions of z, we have here two entire functions of z:

$$\frac{\exp(iz)+\exp(-iz)}{2} \quad \text{and} \quad \frac{\exp(iz)-\exp(-iz)}{2i},$$

which, for real values: $z = x$, take up real values coinciding with $\cos x$ and $\sin x$, respectively. It is, therefore, natural to denote, by definition, the first of these by $\cos z$ and the second by $\sin z$ and to call them the fundamental *trigonometric functions—the cosine and the sine of* z:

$$\cos z = \frac{\exp(iz)+\exp(-iz)}{2} \quad \text{and} \quad \sin z = \frac{\exp(iz)-\exp(-iz)}{2i}. \tag{25}$$

Formulae (25) are called *Euler's formulae*. The formula obtained by multiplying both the sides of the second formula by i and adding termwise to the first formula,

$$\exp(iz) = \cos z + i \sin z, \tag{26}$$

is also called Euler's formula.

It follows directly from (25) that $\cos z$ is an *even* and $\sin z$ is an *odd* function:

$$\cos(-z) = \cos(z) \quad \text{and} \quad \sin(-z) = -\sin z. \tag{27}$$

From the same formulae (25) it follows that $\cos z$ and $\sin z$ have a period 2π (because when z is increased by 2π, the arguments of the exponential functions on the right-hand sides of the formulae change by $\pm 2\pi i$—the period of the exponential function). We shall show that 2π is the *fundamental period* of the functions $\cos z$ and $\sin z$. In fact, if ω is the period of the function $\cos z$, then

$$\cos(z+\omega) = \cos z;$$

moreover, for $z = \pi/2$ we have

$$\cos\left(\omega+\frac{\pi}{2}\right) = 0.$$

But it follows from above that

$$\exp\left[i\left(\omega+\frac{\pi}{2}\right)\right]+\exp\left[-i\left(\omega+\frac{\pi}{2}\right)\right]=0,$$

or

$$\exp\left[i\,(2\omega+\pi)\right]=-1.$$

Hence, by formula (24)

$$i\,(2\omega+\pi)=\ln|-1|+i\,\mathrm{Arg}\,(-1)=i\,(\pi+2k\pi),$$

that is,

$$\omega=k\pi,$$

and since $\cos\omega=\cos 0=1$, k is an even number, i.e.

$$\omega=2m\pi.$$

Similarly, we can establish that 2π is the *fundamental* period of the function $\sin z$.

We now proceed on to derive the *addition theorem* for the functions $\cos z$ and $\sin z$, i.e. to find the relations between $\cos(z_1+z_2)$ and $\sin(z_1+z_2)$ on the one hand and $\cos z_1$, $\cos z_2$, $\sin z_1$ and $\sin z_2$ on the other (z_1 and z_2 being any complex numbers). We shall obtain these relations as corollaries to the corresponding theorem for the exponential function.

Replacing z by the sum z_1+z_2 in the formula (26), we find

$$\begin{aligned}\cos(z_1+z_2)+i\sin(z_1+z_2)&=\exp\left[i\,(z_1+z_2)\right]=\exp(iz_1)\cdot\exp(iz_2)=\\&=(\cos z_1+i\sin z_1)(\cos z_2+i\sin z_2),\end{aligned}$$

that is,

$$\begin{aligned}&\cos(z_1+z_2)+i\sin(z_1+z_2)=\\&\quad=(\cos z_1\cos z_2-\sin z_1\sin z_2)+i\,(\sin z_1\cos z_2+\cos z_1\sin z_2).\end{aligned}$$

If we put here $-z_1$ and $-z_2$ for z_1 and z_2 respectively and use the relation (27), we get

$$\begin{aligned}&\cos(z_1+z_2)-i\sin(z_1+z_2)=\\&\quad=(\cos z_1\cos z_2-\sin z_1\sin z_2)-i\,(\sin z_1\cos z_2+\cos z_1\sin z_2).\end{aligned}$$

Adding and subtracting the last two formulae termwise, we obtain

$$\left.\begin{aligned}\cos(z_1+z_2)&=\cos z_1\cos z_2-\sin z_1\sin z_2,\\ \sin(z_1+z_2)&=\sin z_1\cos z_2+\cos z_1\sin z_2.\end{aligned}\right\}\qquad(28)$$

These formulae are of fundamental importance in the theory of trigonometric functions. In particular, they include the so-called formulae of *reduced arguments.* In fact, if we put in the formulae (28) $z_1 = z$ and $z_2 = \frac{\pi}{2}$, we get

$$\cos\left(z+\frac{\pi}{2}\right) = \cos z \cos\frac{\pi}{2} - \sin z \sin\frac{\pi}{2} = -\sin z,$$

$$\sin\left(z+\frac{\pi}{2}\right) = \sin z \cos\frac{\pi}{2} + \cos z \sin\frac{\pi}{2} = \cos z.$$

For $z_1 = z$ and $z_2 = \pi$, we obtain another pair of reduction formulae:

$$\cos(z+\pi) = -\cos z,$$
$$\sin(z+\pi) = -\sin z$$

and so on.

Putting into the first of the formulae (28) $z_1 = z$ and $z_2 = -z$, we obtain the following relation between sin z and cos z:

$$1 = \cos^2 z + \sin^2 z. \tag{29}$$

We thus find that all the well known formulae or relations among trigonometric functions of real variables are also valid for the complex variables. However, from the formula (29) we cannot now conclude that $|\cos z| \leqslant 1$ and $|\sin z| \leqslant 1$, because $\cos^2 z$ and $\sin^2 z$ are not, generally, real nonnegative numbers.

The *hyperbolic functions* cosh z and sinh z defined by the formulae

$$\cosh z = \frac{\exp z + \exp(-z)}{2} \quad \text{and} \quad \sinh z = \frac{\exp z - \exp(-z)}{2} \tag{30}$$

are intimately connected with the trigonometric functions cos z and sin z.

When z is real and equal to x, these functions, obviously, take up real values and become the functions cosh x and sinh x, well known in analysis. The first of them (which is even) decreases in the half-interval $-\infty < x \leqslant 0$ from ∞ to unity and thereafter increases from unity to ∞ in the half-interval $0 \leqslant x < \infty$. The second one (which is odd) increases, in the entire infinite interval $-\infty < x < +\infty$, from $-\infty$ to $+\infty$ and vanishes at $x = 0$.

Comparing the formula (30) with the formulae (25), it follows that the following relations exist between the triogonometric and the hyperbolic

functions:

$$\cosh z = \cos (iz) \quad \text{and} \quad \sinh z = -i \sin (iz). \tag{31}$$

From these it follows, in particular, that

$$\cosh^2 z - \sinh^2 z = [\cos (iz)]^2 + [\sin (iz)]^2 = 1. \tag{32}$$

We shall now determine the real and imaginary parts and also the moduli of the functions $\cos z$ and $\sin z$. Putting $z = x + iy$, we obtain from (28) and (30)

$$\left.\begin{aligned} \cos (x+iy) &= \cos x \cos (iy) - \sin x \sin (iy) = \cos x \cosh y - i \sin x \sinh y, \\ \sin (x+iy) &= \sin x \cos (iy) + \cos x \sin (iy) = \sin x \cosh y + i \cos x \sinh y. \end{aligned}\right\} \tag{33}$$

Hence

$$\left.\begin{aligned} \operatorname{Re} \cos (x+iy) &= \cos x \cosh y, & \operatorname{Im} \cos (x+iy) &= -\sin x \sinh y, \\ \operatorname{Re} \sin (x+iy) &= \sin x \cosh y, & \operatorname{Im} \sin (x+iy) &= \cos x \sinh y. \end{aligned}\right\} \tag{34}$$

For the moduli of the functions $\cos z$ and $\sin z$, we obtain

$$|\cos z| = \sqrt[+]{(\cos x \cosh y)^2 + (\sin x \sinh y)^2} =$$

$$= \sqrt[+]{\cosh^2 y (1 - \sin^2 x) + \sin^2 x \sinh^2 y} = \sqrt[+]{\cosh^2 y - \sin^2 x}$$

and similarly $|\sin z| = \sqrt[+]{\sinh^2 y + \sin^2 x}$. Thus,

$$|\cos z| = \sqrt[+]{\cosh^2 y - \sin^2 x}, \quad |\sin z| = \sqrt[+]{\sinh^2 y + \sin^2 x}. \tag{35}$$

From these relations follow the inequalities

$$\left.\begin{aligned} &\cosh y \geqslant |\cos z| \geqslant \sqrt[+]{\cosh^2 y - 1} = |\sinh y|, \\ &\sqrt[+]{\sinh^2 y + 1} = \cosh y \geqslant |\sin z| \geqslant |\sinh y|. \end{aligned}\right\} \tag{36}$$

We find that the *moduli of the functions* $\cos z$ *and* $\sin z$ *increase, without limit, along with* $|y|$ *as we recede from the real axis, this increase being no faster than the increase of* $\cosh y$ *and no slower than that of* $\sinh y$. Fig. 21 represents the surface $u = |\sin z|$, the so-called *sine relief*.[1] Since $\sinh y \neq 0$ if $y \neq 0$, it further follows from the inequalities (36) that $\cos z$ and $\sin z$ cannot vanish outside the real axis, i.e. the equations $\cos z = 0$ and $\sin z = 0$ do not have imaginary roots. Consequently, all

[1] The figure has been taken from *Tables of Functions*, Jahnke and Emde, Gostekhizdat, 1948.

the roots of these equations reduce to roots that are well known in trigonometry

$$z = (2k-1)\frac{\pi}{2} \quad \text{for the equation } \cos z = 0$$

and

$$z = k\pi \qquad \text{for the equation } \sin z = 0.$$

We also note down the formulae for the derivatives of the trigonometric and hyperbolic functions :

$$(\cos z)' = \left[\frac{\exp(iz)+\exp(-iz)}{2}\right]' = i\,\frac{\exp(iz)-\exp(-iz)}{2} = -\sin z,$$

$$(\sin z)' = \cos z, \qquad (\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z.$$

On the basis of the functions $\sin z$ and $\cos z$ one may also define and

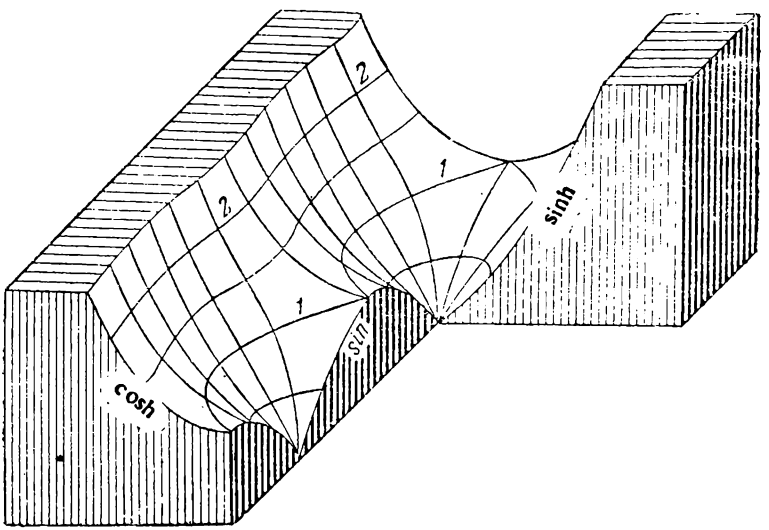


Fig. 21.

study various other trigonometric functions of the complex variable, such as

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

14. GEOMETRICAL BEHAVIOUR

We now study the geometrical behaviour of the trigonometric functions. For this purpose the study may be confined to the mapping

$$w = \cos z,$$

because the mapping $w = \sin z$ can be represented in the form

$$w = -\cos\left(z + \frac{\pi}{2}\right)$$

and, consequently, reduced to the combination of the translation $z_1 = z + \pi/2$ of the z-plane in the direction of the real axis, the mapping $z_2 = \cos z_1$, and finally $w = -z_2$, i.e. a rotation of the whole plane, about the origin of coordinates, through an angle π.

We shall first examine the pre-images of the point w under the mapping $w = \cos z$, i.e. the roots of the equation

$$w = \cos z, \tag{37}$$

where w is any arbitrary finite complex number. Replacing $\cos z$ by the Euler formula (25) and assuming, for the sake of brevity,

$$\exp(iz) = t, \tag{38}$$

we obtain for t the equation

$$w = \frac{t+t^{-1}}{2},$$

that is,

$$t^2 - 2wt + 1 = 0, \tag{39}$$

whence we have

$$t_j = w + \sqrt{w^2-1} \qquad (j=1, 2) \tag{40}$$

[we shall not put the double sign before the square root because this root by itself has got two values]. Evidently, the product of the numbers t_1 and t_2 is equal to unity and hence each of them is nonvanishing. Denoting one of them by τ and the other by $\frac{1}{\tau}$, we obtain from (38) two equations for determining z:

$$\exp(iz) = \tau\ (\neq 0) \quad \text{and} \quad \exp(iz) = \frac{1}{\tau}\ (\neq 0). \tag{41}$$

Each of these equations, as shown in Art. 12, has an infinite set of solutions expressed in the formula (24):

$$iz' = \ln|\tau| + i\operatorname{Arg}\tau \quad \text{and} \quad iz'' = \ln\left|\frac{1}{\tau}\right| + i\operatorname{Arg}\frac{1}{\tau} = -(\ln|\tau| + i\operatorname{Arg}\tau),$$

or

$$z' = \operatorname{Arg} \tau - i \ln |\tau| \quad \text{and} \quad z'' = -(\operatorname{Arg} \tau - i \ln |\tau|). \tag{42}$$

We obtain two infinite point sets located on the pair of straight lines $y = \pm \ln |\tau|$ parallel to the real axis. On one of them the neighbouring points z' and on the other the neighbouring points z'' lie at a distance of 2π from one another and for each point z' lying on the straight line $y = -\ln |\tau|$, there is a point z'' lying on the other, symmetric to z' with respect to the origin of coordinates (vide Fig. 22, corresponding to the case $|\tau| < 1$). For $w = \pm 1$, the roots τ and $\frac{1}{\tau}$ of the equation (39) are equal to ± 1. Then, both the straight lines coincide with the real axis and the two point sets z' and z'' also coincide.

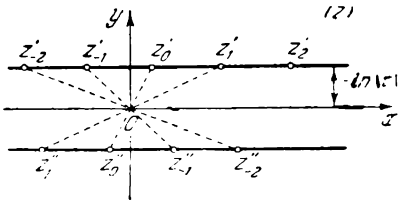


Fig. 22

Thus, in all cases, equation (37) has solutions which form an infinite set. Hence, it follows that firstly the function $w = \cos z$ maps the finite z-plane onto the whole (finite) w-plane and secondly every point w has an infinite set of pre-images in the z-plane. This mapping is conformal at all those points at which $(\cos z)' = -\sin z \neq 0$, i.e. for $z \neq k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$).

Let z describe a straight line parallel to one of the coordinate axes. If this be the straight line $z = c + it$, parallel to the imaginary axis, its image would be the curve L:

$$w = \cos z = \cos c \cosh t - i \sin c \sinh t$$

[vide first of the formulas (33)]. When $c = k\pi$ we obtain $w = \cos k\pi \cosh t = (-1)^k \cosh t$ $(-\infty < t < +\infty)$, i.e. w describes twice the part $u \geqslant 1$ of the real axis when k is even and the part $u \leqslant -1$ when k is odd. When $c = (2k-1)\,\pi/2$ we obtain $w = (-1)^k i \sinh t$, i.e. w describes the whole imaginary axis once in the direction of increasing u when k is even and in the direction of decreasing u when k is odd.

Now, let $c \neq m\,\pi/2$, m being an integer. We rewrite the equation of the curve L in the form

$$u = \cos c \cdot \cosh t \quad \text{and} \quad v = -\sin c \cdot \sinh t \quad (-\infty < t < \infty) \tag{43}$$

or, eliminating the parameter t ($\cos c \neq 0$ and $\sin c \neq 0$),

$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 1. \tag{44}$$

We thus obtain the equation of a hyperbola with semi-axes $|\cos c|$ and $|\sin c|$ and with foci at the points ± 1.

However, we must not imagine that the curve L coincides with the whole hyperbola. It follows from the parametric representation (43) of L that u preserves always the same sign as that of the expression $\cos c$, while v changes monotonically and continuously from $-\infty$ to $+\infty$ (and conversely). Hence it follows that the curve L coincides only with one of

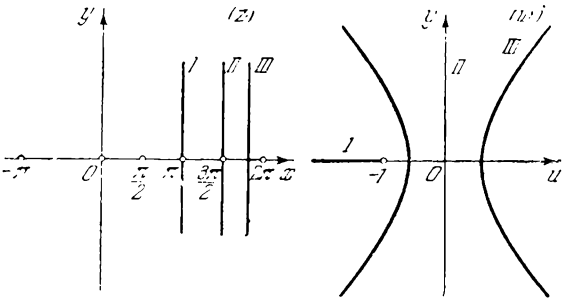


Fig. 23.

the two branches of the hyperbola (43), namely with the right branch if $\cos c > 0$ and with the left if $\cos c < 0$. On the right half of the Fig. 23 the images of the three straight lines of the z-plane

$$\text{I } (x=\pi), \quad \text{II } \left(x=\frac{3\pi}{2}\right) \text{ and III } \left(x=c, \text{ where } \frac{3\pi}{2}<c<2\pi\right)$$

have been shown diagrammatically.

The mapping of the straight line $z = c + it$ onto the corresponding branch is here one-to-one and each of the two half-straight lines, into which the straight line in question is divided by the real axis, is mapped one-to-one onto one of the half-branches into which the hyperbola is split by the vertex.

Now let z describe the straight line l' : $z = t + ic'$, parallel to the real axis. Its image will be the curve L' :

$$w = \cos z = \cos t \cosh c' - i \sin t \sinh c'.$$

When $c' = 0$, l' is the real axis and L' has the equation $w = \cos t$ $(-\infty<t<+\infty)$; consequently, w describes the segment $-1 \leqslant u \leqslant 1$ of the real axis infinitely many times, and to every section of the straight line l', of length 2π, there corresponds a double description of the above-mentioned segment. Let $c' \neq 0$. Then, we may write the equation of the curve L' in the form

$$u = \cos t \cosh c' \quad \text{and} \quad v = -\sin t \sinh c' \quad (-\infty<t<\infty) \qquad (45)$$

whence, on eliminating the parameter t ($\cosh c' \neq 0$, $\sinh c' \neq 0$), we get

$$\frac{u^2}{\cosh^2 c'} + \frac{v^2}{\sinh^2 c'} = 1. \tag{46}$$

This is the equation of an ellipse with semiaxes $|\cosh c'|$ and $|\sinh c'|$ and foci at the points ± 1. From the parametric form (45) of the curve L', it follows that the point w describes the ellipse in one and the same direction an infinite number of times, each description corresponding to a shift of the point z along the straight line $z = t + ic'$ by 2π (Fig. 24 gives on the right-hand side the images of the two straight lines of the z-plane, I ($y=0$) and II ($y=c\neq 0$). Thus, *the mapping*

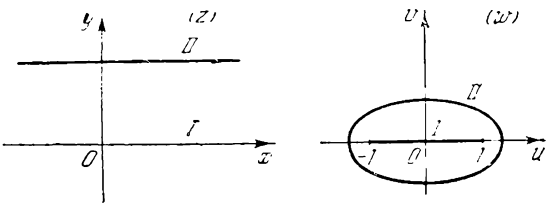


Fig. 24.

$w = \cos z$ *transforms the orthogonal network of straight lines parallel to the coordinate axes into the network of ellipses and hyperbolae with common foci at* ± 1. Since the mapping is conformal at all points of the z-plane, excepting those of the form $z = k\pi$ ($k=0, \pm 1, \pm 2, \ldots$) [images of which are just the above-mentioned foci], the network of confocal ellipses and hyperbolae must also be orthogonal. We have already come across this result in Art. 10.

15. CONTINUATION

In the z-plane we take a domain g which might be mapped by the function $w = \cos z$ one-to-one onto the corresponding domain of the w-plane. This domain might be selected in various ways. We should only be careful to ensure that no two pre-images of one and the same point w belong to it. We choose, for example, a half-strip with width h ($0 < h \leqslant 2\pi$), parallel to the imaginary axis, with its base located on the real axis (Fig. 25). Evidently, it satisfies the conditions imposed. In fact, if for any point $z_0 \in g$, $\cos z_0 = w_0$, then all the other pre-images of the point w_0 in the z-plane must, as we have already seen (page 98), lie partly on a straight line parallel to the real axis and passing through the point z_0 and partly on a straight line symmetric to the first with respect to the real axis. But the pre-images located on the first straight line will lie at distances which are a multiple of 2π from the point z_0. Since the

width of the half-strip is not greater than 2π, none of the points falls within the interior of the half-strip or on its boundary. The second straight line has no point in common with the half-strip. Thus, the function $w = \cos z$ maps the domain g, one-to-one and conformally, onto a certain point set in the w-plane.

In order to construct this set, we make the point z describe the boundary γ of the domain g such that it continuously describes first the side I of the half-strip, then the base II and lastly the other side III of the half-strip. In that case, the point $w = \cos z$ also describes successively and continuously the semi-branch (I′) of one hyperbola, the part (II′) which is the image of real axis and is represented by the segment $-1 \leqslant u \leqslant 1$, $v = 0$ (for the base of the half-strip is not longer than 2π and hence w passes over the last segment no more than twice) and finally the semibranch (III′) of a certain hyperbola.

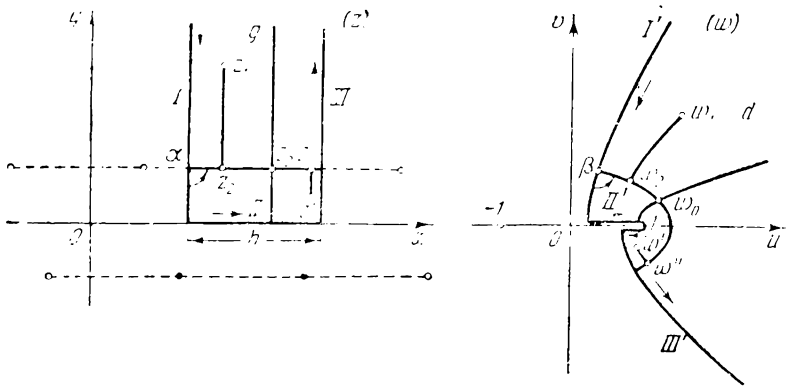


Fig. 25.

The full image of the boundary of the domain g, as obtained in the w-plane (we shall denote it by Γ), divides the plane into two domains. We shall say that one of them is the unknown image d of the domain g (*cf.* Art. 4 of Chapter 1). We shall give two general methods with the help of which we can find out as to which of the domains obtained here represents the image of the domain g.

In the first method, used in Art. 5 of this chapter, any point $z_0 \in g$ is chosen and its image $w_0 = \cos z_0$ is marked. This image cannot belong to the contour Γ because, otherwise, one of the pre-images of the point w_0 would belong to the domain g and the other to the boundary of this domain, which, as we have seen, is impossible. Hence, the point w_0 falls into one of the above-mentioned domains. This would really be the domain sought for.

In the other method, used in Art. 7 of this chapter, we mark the

direction of describing the boundary γ of the domain g. This direction can be followed, for example, by imagining an observer moving with the point z along the boundary of the domain g and noting on which side the interior of the domain lies. In the description, as shown in our figure, the domain g remains, evidently, to the left of the observer. We now make the same observer move with the point $w = \cos z$ along Γ. Then, he too must observe the image of the domain g on the same side, i.e. to his left.

We finally note that the form of the domain d will generally change with a variation in the position and width of the half-strip g. In Fig. 26, we have represented the case when the base of the strip belongs to one of the intervals of the form $[k\pi, (k+1)\pi]$. On the other hand, the case represented in Fig. 25 and characterised by the fact that the base of the strip were, so to say, fractured as a result of mapping at the point $w = 1$ or $w = -1$, is met with when the base of the half-strip contains in it a point of the form $k\pi$.

As an exercise we leave it to the reader to study the mapping of the strip $0 < x < \pi$ by the function $w = \cos z$, looking upon this mapping as composed of the mappings

$$z_1 = iz, \quad z_2 = e^{z_1}, \quad w = z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right),$$

effected successively.

As a result of this the reader should find that $w = \cos z$ maps the strip in question, one-to-one and conformally, onto a domain of the w-plane bounded by the infinite segment of the real axis connecting the points -1 and $+1$ through the point at infinity.

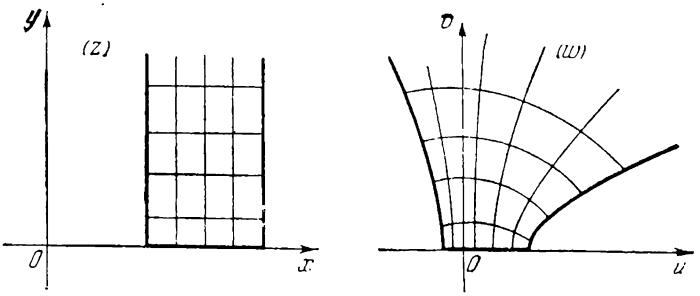


Fig. 26.

16. ONE-VALUED BRANCHES OF THE MANY-VALUED FUNCTIONS

The functions $w = f(z)$ studied in the previous sections generally take up one and the same value w at several (two or more) points of the z-plane. The only exception to this was the bilinear function which

gave a one-to-one mapping of the extended plane onto itself. If we leave out this exception, then, in all the remaining cases, the inverse mapping $z = f^{-1}(w)$ is not one-to-one. This means that functions inverse to those examined above are many-valued.

In order to apply the concepts and results obtained for one-valued functions to the many-valued ones we have to determine the *one-valued branches of these functions.*

This is how these are usually found out. Let $z = f(w)$ be a one-valued and (in the generalized sense) continuous function, defined in the extended plane. We assume that the domain G can be divided by some method into a finite and denumerable set of domains $g_1, g_2, \ldots$, every pair of which has no common points, so that any point of the domain G is an interior point of only one domain g_k or at most a point on the boundary common to two domains g_l and g_k in each of which the mapping $z = f(w)$ is one-to-one. Then, as we already know (Art. 4, Chapter 2), the image of each of these domains g_k will also be a domain $f(g_k) = G_k$, and the whole of the image $f(G)$ will be covered by the domains G_k and also by the images of the common parts of the boundaries of these domains.

We shall examine the inverse function $w = F(z)$ in each of the domains G_k, defining it by the additional condition that its value belongs to g_k, the pre-image of the domain G_k. Then, the function $F(z)$, generally many-valued, is represented by several, or sometimes by an infinitely many, one-valued and (in the generalized sense) continuous functions $F_k(z)$. We shall call each one of these a *one-valued branch* of the function $F(z)$ in the corresponding domain G_k. In this definition it is important to note that the character of the domains G_k and also of the one-valued branches $F_k(z)$ of the function actually depends on how the domain G is divided into the domains g_k. In the simplest cases the domain G admits of a division into domains g_k such that in this division some of the corresponding domains G_k coincide. Let, for instance, $G_{k_1}, G_{k_2}, \ldots$, coincide with one and the same domain G'. Then, the many-valued function $w = F(z)$ possesses several, sometimes infinitely many, one-valued branches in the domain G', namely the branches $F_{k_1}(z)$, $F_{k_2}(z), \ldots$.

We should add here that for any arbitrary continuous function $z = f(w)$ a division of the domain G into the domains g_k, satisfying the foregoing conditions, is, generally speaking, impossible. However, when $f(w)$ is analytic in the domain G (excluding the isolated points at which it becomes ∞), such a division is always feasible and can be effected in infinitely many ways.

We call the function $z = f(w)$, analytic in some domain g (excluding the points at which it becomes ∞) and taking up different values [$f(w_1) \neq f(w_2)$ if $w_1 \neq w_2$ and $w_1, w_2 \in g$], *single-sheeted* in the domain g. If, however, there exists in the domain at least one pair of different points at which $f(w)$ takes up one and the same value, i.e. $f(w_1) = f(w_2)$, $w_1 \neq w_2$, we call the function *multi-sheeted* in this domain.

The fact stated above may (without proof) be formulated as follows. If the analytic function $z = f(w)$ is multi-sheeted in the domain G, this domain can be divided into a finite and denumerable set of domains, in each one of which $f(w)$ is single-sheeted. The corresponding domains g_k are called *domains of single-sheetedness of the function* $f(w)$.

Thus, the method of finding out one-valued branches is always applicable to the functions which are inverse to the multi-sheeted ones.

We shall illustrate the above-mentioned method in the case of certain elementary functions. The division of the domain G into single-sheeted domains can always be effected by using well known properties of elementary functions.

Besides the functions that are inverse to the elementary functions we shall also examine here other many-valued functions obtained as composite functions, in the form $\varphi\psi(z)$ [where $\varphi(z)$ or $\psi(z)$ is a function inverse to an elementary function] or as rational combinations of elementary functions.

17. THE FUNCTION $w = \sqrt[n]{z}$.

We shall now examine the radical $w = \sqrt[n]{z}$, representing a function inverse to the power function $z = w^n$ (n being a natural number greater than unity).

For any z, finite and nonvanishing, the radical has n different values given by the formula

$$w = \sqrt[n]{|z|}\left(\cos\frac{\operatorname{Arg} z}{n} + i\sin\frac{\operatorname{Arg} z}{n}\right). \tag{47}$$

When $z = 0$ or $z = \infty$ we obtain one-valued functions, $w = 0$ or $w = \infty$, respectively.

The n values (47), representing those points of the w-plane at which w^n take up one and the same value of z, are located at the vertices of a regular polygon inscribed in the circle $|w| = \sqrt[n]{|z|}$.

Conversely, the vertices of any regular polygon with centre at the origin of coordinates can be looked upon as n values of $\sqrt[n]{z}$. Hence, the domain g in the w-plane will be a domain single-sheeted for $z = w^n$ if, and only if, it does not contain more than one of the n vertices of any regular polygon with centre at $w = 0$.

This condition is, evidently, satisfied by every angle of aperture $2\pi/n$ with its vertex at the origin of coordinates.

We draw, from the origin of coordinates, n rectilinear rays at equal intervals of angles. We shall then find that all planes in which the multi-sheeted function $z = w^n$ is defined can be divided into n-single-sheeted domains of this function : $g_1, g_2, \ldots, g_n$. The image of each one of them will be one and the same domain G' in the z-plane, with some rectilinear ray L emanating from the origin as its boundary. If the domain g_k is bounded by rays making angles $\varphi_0 + \frac{2k\pi}{n}$ and $\varphi_0 + \frac{2(k+1)\pi}{n}$ with the positive direction of the real axis, then the ray L makes an angle $n\varphi_0$ with the positive direction of the real axis.

In conformity with what has been stated in Art. 16 we obtain in the domain G', n one-valued branches of the function $\sqrt[n]{z}$. Each of them; $\sqrt[n]{z}$ $(k=1, 2, \ldots, n)$, is fully determined by the condition that its value $w = \sqrt[n]{z}$ belongs to the domain g_k. Since $z = w^n$ has a nonzero derivative at all points of the domain g_k : $z' = nw^{n-1}$, the branches $\sqrt[n]{z}$ also possess derivatives

$$(\sqrt[n]{z})' = \frac{1}{nw^{n-1}} = \frac{1}{n(\sqrt[n]{z})^{n-1}},$$

which are nonvanishing.

We now take a system of rectilinear rays emanating from the origin of

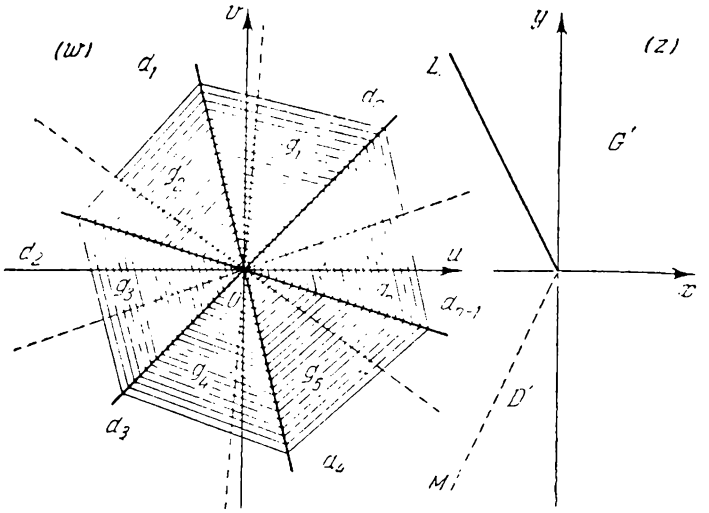


Fig. 27.

coordinates obtained by the above method of rotation about the origin of

coordinates through an angle α $(0<\alpha<2\pi/n)$. Then, the new system divides the w-plane into n domains $d_1, \ldots, d_n$, of which every domain d_k possesses parts in common with two neighbouring domains g_k and g_{k+1} [if $k = n$, then g_{n+1} has to be replaced by g_1] (Fig. 27).

The image of each one of the domains d_k in the w-plane will be one and the same domain D bounded by the rectilinear ray M emanating from the origin of coordinates and making an angle $n\varphi_0 + n\alpha$ with the positive direction of the real axis. In this domain we also get n one-valued branches of the function $\sqrt[n]{z}$, each one of which is fully determined by the fact that its values belong to the corresponding domain d_k. We denote these branches by $(\sqrt[n]{z})_k$. These are differentiable in D', with derivatives

$$(\sqrt[n]{z})_k' = 1 : n\,(\sqrt[n]{z})_k^{n-1}.$$

We compare them with the branches $\sqrt[n]{z}$. Since the pre-image d_k of the domain D' in the w-plane belongs partly to the domain g_k and partly to the domain g_{k+1}, the branch $(\sqrt[n]{z})_k$ in one part of the domain D' (representing the image of the part common to the domains d_k and g_k) will coincide with $\sqrt[n]{z}$, while in the other part of the domain D' (representing the image of the part common to the domains d_k and g_{k+1}) will coincide with $\sqrt[n]{z}$.

We find that if certain domains of single-sheetedness are replaced by others the domain of every new one-valued branch will be obtained by joining a part of the domain of definition of one of the previous branches with a part of the domain of definition of another.

If the angle of rotation $\alpha = 0$, then d_k coincides with g_k, D' with G' and each branch $(\sqrt[n]{z})_k$ with $\sqrt[n]{z}$. When, however, the angle α, increasing continuously, approaches $2\pi/n$, then the domain d_k tends to g_{k+1}, the corresponding domain D' to G' and the branch $(\sqrt[n]{z})_k$ to the branch $\sqrt[n]{z}$ (instead of g_{n+1} and $\sqrt[n]{z}$ we should take g_1 and $\sqrt[n]{z}$) in a larger and larger part of the domain D'. When $\alpha = 2\pi/n$, d_k coincides with g_{k+1}, D' with G' and the branch $(\sqrt[n]{z})_k$ passes over to the branch $\sqrt[n]{z}$.

The transition of one branch, $\sqrt[n]{z}$, into the other, $\sqrt[n]{z}$, may be followed up by making the point z describe a full circle, with centre at the origin of coordinates. If the value of $\sqrt[n]{z}$ at the point z_0 were taken to be one belonging to the branch $\sqrt[n]{z}$ and were represented by the point

w_0 in the domain g_k :

$$w_0 = \sqrt[n]{|z_0|}\left(\cos\frac{\varphi_0}{n} + i\sin\frac{\varphi_0}{n}\right),$$

then, for the point z moving continuously along the circle $|z| = |z_0|$ in the positive direction, the corresponding values of the radical

$$w = \sqrt[n]{|z_0|}\left(\cos\frac{\varphi_0}{n} + i\sin\frac{\varphi}{n}\right)$$

would vary continuously with φ, and after one full description (with return of the point z to the original position z_0) the value of the radical would pass over to

$$w_1 = \sqrt[n]{z_0}\left(\cos\frac{\varphi_0+2\pi}{n} + i\sin\frac{\varphi_0+2\pi}{n}\right).$$

This last result is obtained from w_0 by a rotation about the origin of coordinates through an angle $2\pi/n$. Hence, the point w_1 belongs to the domain g_{k+1}, adjacent to g_k, and w_1 is the value of the branch $\sqrt[n]{z}_{k+1}$ at the point z_0.

This conclusion is applicable to any point of the domain G'. From this it follows that as the point z describes a circle of any radius, with its centre at the origin of coordinates, the values of $\sqrt[n]{z}$, changing continuously, pass over from the branch $\sqrt[n]{z}_k$ to the branch $\sqrt[n]{z}_{k+1}$.

It is necessary for the point z to traverse n times around the point $z = 0$ in the positive direction, so that the branches of the radical $\sqrt[n]{z}$, being replaced one by another ($\sqrt[n]{z}_k$ by $\sqrt[n]{z}_{k+1}$, $\sqrt[n]{z}_{k+1}$ by $\sqrt[n]{z}_{k+2}$, . . . , $\sqrt[n]{z}_n$ by $\sqrt[n]{z}_1$, . . . , $\sqrt[n]{z}_{k-1}$ by $\sqrt[n]{z}_k$) come once again over to themselves.

The point possessing the property that a full (single) revolution around it, in any of its neighbourhoods and along any closed Jordan curve, replaces one continuously varying branch of a many-valued function by another branch of the same function, is called the *point of branching* (or *the branch point*) of the function.

The fact that after n revolutions in one and the same direction we come back to the original branch means, so to say, that the given branch point possesses a *finite* order, viz. the order $n - 1$, and this point is called an *algebraic branch point*.[1]

Thus, the point $z = 0$ is an algebraic branch point of order $n - 1$ for the function $\sqrt[n]{z}$.

[1] In this concept, it is also assumed that there exists a (finite or infinite) limit of the function at the given point.

Evidently, the point $z = \infty$ may also be looked upon as an algebraic branch point of order $n - 1$ of the function $\sqrt[n]{z}$ because each revolution around it along a circle of any radius (however large) with centre at the origin of coordinates is also a revolution around the origin of coordinates. Hence, the many-valued function $w = \sqrt[n]{z}$ has two branch points in the z-plane, viz. $z = 0$ and $z = \infty$, both of order $n - 1$.

The one-valued branches of this function, as described above, have been constructed for the G' or D' type of domains, the boundary of which is represented by a rectilinear ray joining both the branch points. A more general type of a similar domain is obtained if instead of the rectilinear ray we draw in the extended plane any Jordan curve joining the points 0 and ∞. Let Γ be such a curve and G the domain bounded by it. If z describes the curve Γ from the initial point 0 to the terminal point ∞, then the n points $w = \sqrt[n]{z}$ corresponding to z will describe n Jordan curves γ_k connecting the point 0 with the point ∞. These curves have no other common points except 0 and ∞ and they form *pairwise* (γ_k with γ_{k+1}) closed Jordan curves of the extended plane.

Of the two domains bounded by the curves γ_k and γ_{k+1}, let g'_k be the one that does not contain the curves $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_n$. If the z-plane be rotated about the origin of coordinates through an angle $2\pi/n$, then by construction γ_k goes over to γ_{k+1}, γ_{k+1} to γ_{k+2} and the domain g'_k to the domain g'_{k+1}. Since g'_k and g'_{k+1} have no points in common, none of them would contain any such pair of points as may traverse one into the other as a result of this rotation. Hence, all the domains g'_k ($k=1, 2, \ldots, n$) are domains of single-sheetedness of the function $z = w^n$ and we obtain n one-valued branches of the function $\sqrt[n]{z}$ in the domain G if we impose the requirement that the values of any particular branch would lie in the corresponding domain g'_k. For fixing a branch it is sufficient to indicate the value of $\sqrt[n]{z}$ at any point z_0 of the domain G. If this value is w_0, we find a unique domain g'_k containing the point w_0 and along with it the unique branch $\sqrt[n]{z}$ in the domain G taking up the value w_0 at the point z_0. We proceed exactly in the same manner if we want to fix a definite branch $\sqrt[n]{z}$ in a G type domain.

Let $\sqrt[n]{z}$ and $\sqrt[n]{z}$ be two branches of $\sqrt[n]{z}$ in the domain G and let w'_0 and w''_0 be their values at a certain point z_0. Since

$$w'_0 = \sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi_0+2m'\pi}{n} + i\sin\frac{\varphi_0+2m'\pi}{n}\right),$$

$$w''_0 = \sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi_0+2m''\pi}{n} + i\sin\frac{\varphi_0+2m''\pi}{n}\right),$$

where m' and m'' are integers, w_0'' can be obtained from w_0' on multiplying it by

$$\eta = \cos\frac{2(m''-m')\pi}{n} + i\sin\frac{2(m''-m')\pi}{n},$$

i. e. by one of the values of $\sqrt[n]{1}$. But, on multiplying the function $\sqrt[n]{z}$ by the number η, we, evidently, obtain the function $\eta\sqrt[n]{z}$ which is one-valued and continuous in the domain G, the values of which represent $\sqrt[n]{z}$ and also belong to the same domain which contains the point $\eta\sqrt[n]{z_0}=\sqrt[n]{z_0}$. Consequently, $\eta\sqrt[n]{z}=\sqrt[n]{z}$, in the whole of the domain G. We thus find that the two branches of $\sqrt[n]{z}$, in one and the same domain G, can be obtained one from the other on multiplying the latter by some value of $\sqrt[n]{1}$.

All the results of this section can be extended, with suitable adaptations, to functions of a somewhat more general type

$$w = \sqrt[n]{z-a} \quad \text{or} \quad w = \sqrt[n]{\frac{z-a}{z-b}}.$$

We recommend to the reader to study these cases, bearing in mind that these are functions inverse to the functions $z = a + w^n$ and $z = \frac{bw^n-a}{w^n-1}$, for which the single-sheeted domains are the same as those for the function $z = w^n$. Here, we find that the branch points of the function $w = \sqrt[n]{z-a}$ are a and ∞ and those of the function $w = \sqrt[n]{\frac{z-a}{z-b}}$ are a and b and that it is possible to separate out one-valued branches of the function in any domain whose boundary is a Jordan curve joining the branch points.

18. THE FUNCTION $w = \sqrt[n]{P(z)}$.

For a better exposition of the concept of a branch point we shall examine the many-valued function

$$w = f(z) = \sqrt[n]{P(z)} \tag{48}$$

where $P(z)$ is any polynomial. Let N be the degree of this polynomial; $a_1, a_2, \ldots, a_m$, are its different zeros and $\alpha_1, \alpha_2, \ldots, \alpha_m$ the multiplicities of these zeros ($\alpha_1+\alpha_2+\ldots+\alpha_m=N$). Then, $P(z)$ can be written in the form

$$P(z) = A(z-a_1)^{\alpha_1}\ldots(z-a_m)^{\alpha_m},$$

whence we have

$$f(z) = \sqrt[n]{A\,(z-a_1)^{\alpha_1} \ldots (z-a_m)^{\alpha_m}}. \tag{49}$$

We now examine any closed Jordan curve γ (for instance, a circle) not passing through any of the points a_k ($k = 1, \ldots, m$). We let z describe this curve once in a definite direction. We fix the values of the arguments for the vectors $z - a_1, \ldots, z - a_m$ at any point z_0 on the curve γ. Let these values be $\varphi_1^{(0)}, \ldots, \varphi_m^{(0)}$. When the point z makes a round along the curve γ the angles φ_k between the vectros $z - a_k$ and the positive direction of the real axis will change continuously, starting from the initial values $\varphi_k^{(0)}$, and as a result of making one round along the curve γ they either return to the initial values $\varphi_k^{(0)}$ (if the point a_k lies in the exterior of γ) or get increased by $\pm 2\pi$ (if the point lies in the interior of γ_k).[1]

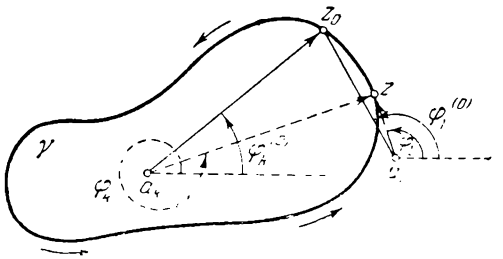


Fig. 28

Here, the sign, + or −, of the increment will depend only on the direction chosen for the description of the curve γ. We shall call that direction as positive for which the increment of the angle is equal to plus 2π. For this we shall assume that the point z describes γ in the positive direction. If none of the points a_k lies in the interior of γ, then all the angles φ_k return, as a result of the description, to the initial values $\varphi_k^{(0)}$ and so the function (49) also returns to its initial value. Hence, it follows that none of the finite points ζ of the plane, different from a_k, can be a branch point of this function. In fact, for any such point we can assign a neighbourhood containing none of the points a_k. Then a round along any closed Jordan curve γ belonging to this neighbourhood and containing the point ζ within its interior will preserve the chosen branch of our function.

Thus, *no finite point ζ different from all a_k can be a branch point of $f(z)$.*

We shall now examine a neighbourhood, of a certain point a_k, so small that it does not contain other points $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m$.

[1] These facts which can be easily proved in very simple cases (for example, when γ is a circle, an ellipse or a polygon) can also be verified for the most general case. See, in this connection, P. S. Alexandrov, *Combinatorial Topology*, Gostekhizdat, 1947; Chapter II.

Then, if the curve γ belonging to this neighbourhood and containing a_k in its interior be described once, the angle φ_k would change by 2π, while all the angles $\varphi_1, \ldots, \varphi_{k-1}, \ldots, \varphi_{k+1}, \ldots, \varphi_m$ would return to their initial values. Hence, it follows that the argument of the expression under the radical sign in (49) changes by $2\pi\alpha_1$ as a result of the traversal of the curve γ and, consequently, the radical (49) acquires a multiplier $\cos\frac{2\pi\alpha_k}{n} + i\sin\frac{2\pi\alpha_k}{n}$, which, evidently, will be different from unity if, and only if, α_k is not a multiple of n. Thus, every zero a_k of the polynomial $P(z)$, the multiplicity α_k of which is not a multiple of n, is a branch point for the function $\sqrt[n]{P(z)}$. To determine the order of this point, we assume that δ_k ($\delta_k < n$) is the highest common factor of α_k and n. Then, assuming $\alpha_k = \delta_k\alpha_k'$ and $n = \delta_k\nu_k$ ($\nu_k > 1$), we write $\left(\cos\frac{2\pi\alpha_k}{n} + i\sin\frac{2\pi\alpha_k}{n}\right)$ in the form $\left(\cos\frac{2\pi\alpha_k'}{\nu_k} + i\sin\frac{2\pi\alpha_k}{\nu_k}\right)$.

As a result of traversing the curve γ p-times in one and the same direction, the function $f(z)$ acquires a multiplier $\cos\frac{2\pi\alpha_k' p}{\nu_k} + i\sin\frac{2\pi\alpha_k' p}{\nu_k}$, which will be, obviously, equal to unity if, and only if, p is a multiple of ν_k. The corresponding least value of p is ν_k. Hence, it follows that the order of the branch point a_k is $\nu_k - 1$.

We shall finally examine the neighbourhood of the point at infinity, containing none of the points a_k, and consider in this neighbourhood the Jordan curve γ containing all the points a_k in its interior. Then, the exterior of γ will contain the point ∞ and will contain none of the points a_k. Let the curve γ be traversed once. All the angles φ then acquire an increment 2π and hence the argument of the expression under the radical sign in (49) changes by $2\pi(\alpha_1 + \alpha_2 + \ldots + \alpha_m)$; any function $f(z)$ acquires a multiplier

$$\cos\frac{2\pi(\alpha_1 + \ldots + \alpha_m)}{n} + i\sin\frac{2\pi(\alpha_1 + \ldots + \alpha_m)}{n} = \cos\frac{2\pi N}{n} + i\sin\frac{2\pi N}{n}.$$

This will be either equal to unity or different from unity depending on whether N is a multiple of n or not. In the first case, ∞ will not be a branch point of the function $f(z)$, while in the second case it will be. If, in the latter case, δ is the highest common factor of N and n ($\delta < n$) and $n = \delta\nu$, the order of the point at infinity, treated as a branch point, will be $\nu - 1$.

We have noted that when α_k is a multiple of n, a traversal of the

Jordan curve γ, including in its interior the point a_k and none of the remaining points a_j, does not change the value of $f(z)$. Similarly, a traversal of the curve γ containing in its interior all the points a_k does not change the value of $f(z)$, when N is a multiple of n.

In general, let $a_{k_1}, \ldots, a_{k_q}$ be a group of branch points for which the sum $\alpha_{k_1} + \ldots + \alpha_{k_q}$ is a multiple of n. Then a traversal of any closed Jordan curve γ containing the above-mentioned points in its interior but no point different from them, cannot change the values of $f(z)$. Hence, we can isolate one-valued branches of the function $f(z)$ in any domain G containing only such closed Jordan curves in whose interior, either there is no branch point a_k or there are groups of branch points for which the sums of the corresponding numbers α_k are divisible by n.

For this purpose it is sufficient to fix the value w_0 of the function $f(z)$ at one of the points z_0 of this domain. One of the n images $f(G)$ of the domain G in the w-plane will contain the point w_0. Let this image be g_k. Then the one-valued branch of the function $f(z)$ in the domain G is fully determined by the condition that all its values belong to g_k. The value of this branch at any point z_1 of the domain G may be obtained as follows. We connect the point z_0 with the point z_1 by a continuous curve λ belonging to the domain G and traverse this curve from the point z_0 to the point z_1, observing that the corresponding value of $f(z)$ changes continuously from the original value w_0. Then, we arrive at the point z_1 from one of the n values of $f(z)$, and denote this one by w_1. This value depends only on the value w_0 chosen at the point z_0 and on the point z_1 itself, and does not depend upon the choice of the path connecting z_0 and z_1; hence, it represents a one-valued function of z_1 in the domain G. In fact, if γ_2 be any other curve joining z_0 and z_1 in the domain G, then while traversing the closed curve γ made up of γ_1 and γ_2 we first obtain, moving from z_0 to z_1 along γ_1, the value w_1 at the point z_1 and thereafter, moving along γ_2 from the point z_1 to z_0, arrive again at the original value w_0 [for a traversal of the closed curve in the domain G cannot, by assumption, lead to a change in the values of the function $f(z)$]. Hence, it also follows that moving along γ_2 from z_0 to z_1 we shall arrive at the point z_1 with the same value w_1 which we would have along γ_1.

We shall explain what has been said above with the help of examples.

1. $w = \sqrt{(1-z^2)(1-k^2z^2)}$, where $0 < k < 1$. This is a two-valued function with four branch points $\pm 1, \pm \frac{1}{k}$. Here, $N = 4$, a multiple of $n = 2$; hence, ∞ is not a branch point.

Since all the numbers α_k are equal to unity $\left(\pm 1, \pm \frac{1}{k}\right.$ being simple

zeros of the expression under the radical), a traversal along any closed curve γ with only two branch points in its interior does not change the value of the function. Hence, we can separate out its one-valued branches, say in the domain G, which is bounded by the two segments: $-\frac{1}{k} \leqslant x \leqslant -1$ and $1 \leqslant x \leqslant \frac{1}{k}$, or in the domain G' whose boundary consists of the segment $-1 \leqslant x \leqslant 1$ and the infinite segment of the real axis connecting the points $-\frac{1}{k}$ and $\frac{1}{k}$ through the point at infinity (Fig. 29).

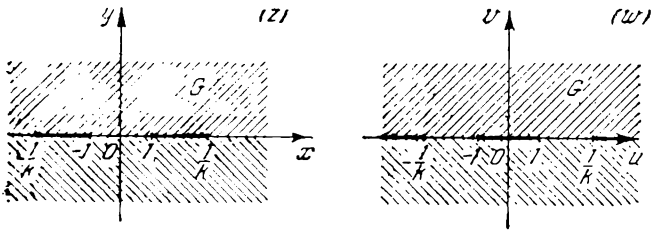


Fig. 29.

In the first of these domains, the branches $f_1(z)$ and $f_2(z)$ can be distinguished by the different values they take up at the origin of coordinates. For example, $f_1(0) = 1$ and $f_2(0) = -1$.

2. $w = \sqrt{4z^3 - g_2 z - g_3}$, where g_2 and g_3 are complex numbers satisfying the condition $g_2^3 - 27g_3^2 \neq 0$, which means that the discriminant of the polynomial $4z^3 - g_2 z - g_3$ does not vanish and hence the zeros e_1, e_2 and e_3 of this polynomial are all different. As, in this example, $N = 3$ is not divisible by $n = 2$, so the point ∞ is also a branch point. Again, a round along a closed Jordan curve containing the pair of branch points in its interior does not change the values of the function. Hence, joining the points e_1 with e_2 and e_3 with ∞ by the Jordan curves γ_1 and γ_2 we obtain a domain G with its boundary consisting of γ_1 and γ_2; in this domain we can isolate the one-valued branches of the given function (Fig. 30).

Fig. 30.

3. We now examine the function inverse to the function $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$, i.e. $w = \varphi(z) = z + \sqrt{z^2 - 1}$. This is a two-valued function, possessing the same branch points as the function $\sqrt{z^2 - 1}$, namely $z = \pm 1$.

To obtain a domain G in which we can isolate the one-valued branches of the function in question, we join the points -1 and $+1$ by a finite segment of the real axis. We then obtain a domain which can be mapped one-to-one by the function $w = z + \sqrt{z^2-1}$ onto any of the two domains: the interior of the unit circle (g_1) or its exterior (g_2) (vide Art. 10). We can isolate any of them by fixing one of the two values of w at any of the points in the domain G, say at the point at infinity. As we find from the formula $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$, z becomes ∞ either for $w = 0$ or for $w = \infty$. Hence, one of the branches of the function $\varphi(z)$ is characterized by the fact that for it $\varphi(\infty) = 0$. This maps G onto the interior of the unit circle. The other branch is characterized by the fact that for it $\varphi(\infty) = \infty$. This branch maps the domain G onto the exterior of the unit circle.

Instead of the domain G we might take the domain, say G', the boundary of which is the infinite segment of the real axis connecting the points -1 and 1, or else the domains G'' and G''' the boundaries of which are the upper and the lower semicircles.

We leave it to the reader to explain, on the basis of the results of Art. 10 the nature of the domain in the w-plane onto which the domain G', G'' or G''', corresponding to the one-valued branches of the function $\varphi(z)$, maps itself.

The whole of the present section pertains to the many-valued function $\sqrt[n]{P(z)}$, where $P(z)$ is a polynomial. The reader can without difficulty extend the results obtained here to a more general function $\sqrt[n]{R(z)}$, where $R(z)$ is any arbitrary rational function. For finding the branch points of the function $\sqrt[n]{R(z)}$ we have to examine all the points of the plane at which $R(z)$ becomes either zero or infinity.

19. LOGARITHM

The function inverse to

$$z = e^w = e^u (\cos v + i \sin v)$$

is defined for any z, finite and nonvanishing, and is represented by the formula [vide formula (24)]

$$w = \ln |z| + i \operatorname{Arg} z.$$

This function, which, is, evidently, many-valued or even infinite-valued, is called the *logarithm* of z and is denoted by $\operatorname{Ln} z$.

Thus, by definition,

$$w = \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z. \tag{50}$$

If we call the value of the logarithm, equal to $\ln |z| + i \operatorname{Arg} z$, the *principal value* and denote it by $\ln z$, then we shall have

$$\operatorname{Ln} z = \ln z + 2k\pi i, \tag{51}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

Hence, it follows that *every complex number which is finite and non-vanishing has an infinite set of logarithms* (i.e. *of values of the logarithmic function*) *which differ from one another by an integral multiple of* $2\pi i$. If z be a real positive number, then the principal value of the logarithm coincides with $\ln |z|$ and, consequently, represents the real number which we come across in our courses on analysis, dealing with logarithms as real functions of a real variable. Thus we obtain: $\ln 1 = 0$, $\ln e = 1$, $\ln 2 = 0.69314718 \ldots$ and so on.

But besides these real values the logarithms of real positive numbers also have an infinite set of complex values obtained from the formula (51). Thus, for instance, $\operatorname{Ln} 1 = 2k\pi i$, $\operatorname{Ln} e = 1 + 2k\pi i$, $\operatorname{Ln} 2 = 0.69314718 \ldots$ $\ldots + 2k\pi i$ and so on.

For real negative numbers and for complex numbers the principal value of the logarithm is

$$\ln |z| + i \arg z \qquad (\arg z \neq 0, \quad |\arg z| \leqslant \pi).$$

All the other values of the logarithm are again complex numbers calculated with the help of the formula (51). For example,

$$\operatorname{Ln}(-1) = (2k+1)\pi i, \qquad \operatorname{Ln}(-2) = 0.69314718 \ldots + (2k+1)\pi i,$$

$$\operatorname{Ln}(1-i) = \ln \sqrt{2} + i\left(-\frac{\pi}{4} + 2\pi k\right) =$$

$$= 0.34657359 \ldots + (8k-1)\frac{\pi i}{4} \quad \text{and so on.}$$

The well known rules for the logarithm of a product or of a quotient of real numbers also hold for the many-valued logarithm of a complex number. Thus

$$\operatorname{Ln}(z_1 \cdot z_2) = \ln |z_1 \cdot z_2| + i \operatorname{Arg}(z_1 \cdot z_2) =$$
$$= \ln |z_1| + \ln |z_2| + i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2, \tag{52}$$

$$\operatorname{Ln} \frac{z_1}{z_2} = \ln \left|\frac{z_1}{z_2}\right| + i \operatorname{Arg} \frac{z_1}{z_2} = \ln |z_1| - \ln |z_2| + i(\operatorname{Arg} z_1 - \operatorname{Arg} z_2) =$$
$$= \operatorname{Ln} z_1 - \operatorname{Ln} z_2. \tag{53}$$

Here, z_1 and z_2 are any complex numbers that do not vanish. In each of these equalities the left- and the right-hand sides for given values of z_1 and z_2 represent infinite sets of complex numbers. The equalities are to be understood in the sense that these sets are identical, i.e. they consist of the same set of elements. A neglect of this fact may lead to errors.

We examine in this connection the following sophism which is due to I. Bernoulli :

$$\operatorname{Ln}(-z) = \operatorname{Ln}(z) \quad \text{for any } z \neq 0.$$

For showing this the following series of equalities is used :

1) $\operatorname{Ln}[(-z)^2] = \operatorname{Ln}(z^2)$, 2) $\operatorname{Ln}(-z) + \operatorname{Ln}(-z) = \operatorname{Ln} z + \operatorname{Ln} z$,
3) $2 \operatorname{Ln}(-z) = 2 \operatorname{Ln} z$ and 4) $\operatorname{Ln}(-z) = \operatorname{Ln} z$.

But an argument like that is wrong, because

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z = \ln |z| + i \arg z + 2k\pi i$$
$$\operatorname{Ln}(-z) = \ln |-z| + i \operatorname{Arg}(-z) = \ln |z| + i \arg z + (2k+1)\pi i,$$

evidently, none of the numbers that are values of Ln z coincides with any of the numbers that are values of Ln $(-z)$.

In the foregoing, error arises while going over from equality (2) to equality (3). The first of them, obtained from the fundamental formula (52), is certainly valid. But the sum $\operatorname{Ln}(-z) + \operatorname{Ln}(-z)$ cannot be replaced by $2 \operatorname{Ln}(-z)$, because the sum in question is obtained from the set of numbers Ln $(-z)$ by adding any of these numbers to the same or a different number belonging to the same set, while, the set $2 \operatorname{Ln}(-z)$ is obtained by doubling every number belonging to the set Ln $(-z)$, i.e. by adding up such a number *only to itself*. Thus, $\operatorname{Ln}(-z) + \operatorname{Ln}(-z) \neq$ $\neq 2 \operatorname{Ln}(-z)$. In exactly the same way

$$\operatorname{Ln} z + \operatorname{Ln} z \neq 2 \operatorname{Ln} z.$$

The reader would understand fully the essence of this exception through a simple example. Let A denote a set consisting of two numbers: 0 and 1. Then $A + A$ denotes the set consisting of three numbers: $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 2$, while the set $2A$ consists of the two numbers $2 \cdot 0 = 0$ and $2 \cdot 1 = 2$.

We further note that putting $z_1 = z_2 = z \neq 0$ in (53) we get

$$\operatorname{Ln} 1 = \operatorname{Ln} z - \operatorname{Ln} z.$$

This correlation is valid, but the right-hand side here should not be replaced by zero, because we are concerned with the set of *all* the differences

between pairs of values of the logarithm of one and the same number. This set consists of all possible integral multiples of the number $2\pi i$, so that we really have

$$\operatorname{Ln} 1 = 2k\pi i \quad (k=0, \pm 1, \pm 2, \ldots).$$

Passing over to the one-valued branches of the logarithm, we first find the domains of single-sheetedness of the function $z = e^w$, the inverse of which is the logarithmic function.

Since all the values of w for which e^w takes up a given value of z ($z \neq 0$ and $z \neq \infty$) are given by the formula (24):

$$w = \ln |z| + i \operatorname{Arg} z$$

and, further, these values are obtained for any of them through a shift by the magnitude $2k\pi i$ ($k=\pm 1, \pm 2, \ldots$), hence the domain of single-sheetedness of the exponential function must not contain any pair of points such that one of the two points of the pair cannot be obtained from the other by such a shift.

All these conditions can be more readily satisfied by taking any rectilinear strip g_0 parallel to the real axis and having a width 2π: $v_0 < v < v_0 + 2\pi$. Besides this strip we obtain an infinite set of domains of single-sheetedness g_k: $v_0 + 2k\pi < v < v_0 + (2k+2)\pi$ ($k = \pm 1, \pm 2, \ldots$).

Obviously, each point of the w-plane will be either an interior point of one of the domains g_k or a common boundary point for two domains g_k and g_{k+1} (Fig. 31). The image of any strip g_k in the z-plane will be one and the same domain G, viz. an angle of aperture 2π with its vertex at the origin of coordinates. The boundary of the domain G will be the rectilinear ray emanating from the origin of coordinates, making an angle v_0 with the real axis.

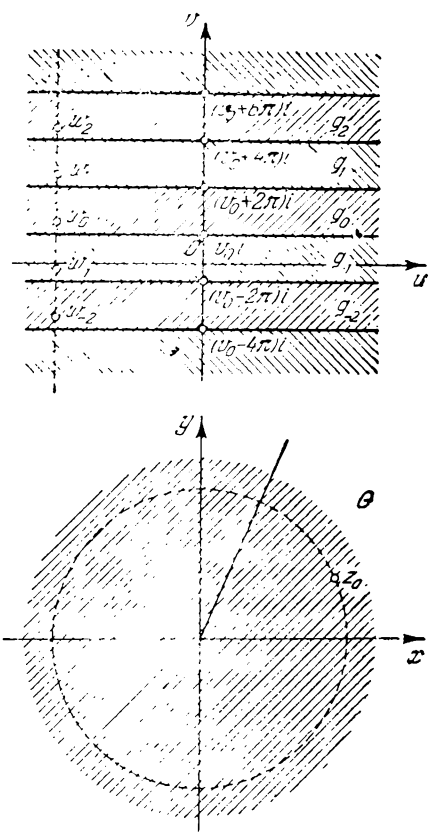


Fig. 31.

In the domain G we obtain an infinite, denumerable set of different one-valued branches of the function Ln z. Each of these branches $\mathrm{Ln}_k z$ will be completely characterized by the fact that its values must belong to a definite strip g_k. However, it will be quite sufficient to fix the value w_0 of the function Ln z at a certain point z_0 of the domain G because one and only one domain (g_{k0}) of all the domains g_k contains the point w_0.

We now examine a branch of the logarithm :

$$\mathrm{Ln}_k z = \ln | z | + i \operatorname{Arg}_k z,$$

where $\operatorname{Arg}_k z$ is the value of the argument, satisfying the condition

$$v_0 + 2k\pi < \operatorname{Arg}_k z < v_0 + (2k+2)\pi.$$

This condition just implies that the values of $\mathrm{Ln}_k z$ belong to the strip g_k.

Since the function $w = \mathrm{Ln}_k z$ effects a one-to-one and continuous mapping of the domain G onto the strip g_k, and the function $z = e^w$ is inverse to it and has a nonvanishing derivative at all points of the domain g_k, the function $\mathrm{Ln}_k z$, by the rule of differentiation of inverse functions, also possesses a derivative given by the formula

$$(\mathrm{Ln}_k z)' = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}.$$

The branch points of the function Ln z are the points zero and infinity. In fact, when z describes once any circle with its centre at the origin of coordinates (however large or small the radius of this circle may be), starting from the initial value $\operatorname{Arg}_k z_0$ of Arg z, changing continuously, acquires on return to the original point an increment of $\pm 2\pi$ (the sign depending upon the sense of description of the circle) ; hence, the branch

$$\mathrm{Ln}_k z = \ln | z | + i \operatorname{Arg}_k z$$

passes over to another branch

$$\mathrm{Ln}_{k\pm 1} z = \ln | z | + i (\operatorname{Arg}_k z \pm 2\pi) = \ln | z | + i \operatorname{Arg}_{k\pm 1} z.$$

Evidently, by describing the circle a number of times in one and the same (say, positive) direction, we would each time obtain a new branch, namely

$$\mathrm{Ln}_{k+1} z, \quad \mathrm{Ln}_{k+2} z, \quad \mathrm{Ln}_{k+3} z, \ldots$$

and would never return to the original branch $\mathrm{Ln}_k z$. Owing to this,

the branch points 0 and ∞ here are called *the branch points of an infinite order* or else *logarithmic branch points.*

Domains of a more general type than G, in which one-valued branches of Ln z might be isolated, can be obtained, for example, if we draw in the z-plane any Jordan curve Γ' connecting the point $z = 0$ with the point $z = \infty$. The images of this curve in the w-plane under the mapping $w = \text{Ln } z$ will be the Jordan curves γ_k' ($k = 0, \pm 1, \pm 2, \ldots$), dividing the w-plane into an infinite set of curvilinear strips g_k' with their boundaries composed of the pairs of curves γ_k' and γ_{k+1}'. In the domain G', with the curve Γ' as its boundary, we shall obtain a denumerable set of one-valued branches of Ln z, viz. $(\text{Ln } z)_k$, each of which maps G' one-to-one onto the corresponding domain g_k'.

Any of the functions $(\text{Ln } z)_k$ can be obtained from any other function $(\text{Ln } z)_m$ by adding to it the corresponding integral multiple of $2\pi i$.

For the derivative of $(\text{Ln } z)_k$, we have the formula :

$$(\text{Ln } z)_k' = \frac{1}{z}.$$

Since this last result is independent of the choice of the branch of Ln z, we can write, in general,

$$(\text{Ln } z)' = \frac{1}{z},$$

understanding the left-hand side as a derivative of any one-valued branch of Ln z, defined in a domain which includes the given point z.

20. THE GENERAL POWER FUNCTION AND EXPONENTIAL FUNCTION

In this section we shall examine the *general power function, the general exponential function and also the logarithm with an arbitrary base.* First of all we must explain the concept of power with an arbitrary index.

Let a be an arbitrary number that does not vanish. If n be an integer, then, as we know, a^n is defined by the relation

$$a^n = |a|^n [\cos (n \text{ Arg } a) + i \sin (n \text{ Arg } a)].$$

If r be any arbitrary rational number equal to p/q, where q is a natural number and the function p/q, is irreducible, then $a^{p/q}$ has q different values, given by the formula (vide. Art. 2 Chapter 1)

$$a^{p/q} = |a|^{p/q}_{+} \left[\cos \left(\frac{p}{q} \text{ Arg } a\right) + i \sin \left(\frac{p}{q} \text{ Arg } a\right)\right].$$

This formula also includes the case of an integral index.

Now, let ρ be a real irrational number. We fix an arbitrary value $\varphi = \operatorname{Arg} a$ and examine a sequence of rational numbers r_n, converging to ρ. A sequence of given values of a^{r_n},

$$|a|_+^{r_n} [\cos (r_n \operatorname{Arg} a) + i \sin (r_n \operatorname{Arg} a)],$$

evidently converges to a definite limit

$$|a|_+^{\rho} [\cos (\rho \operatorname{Arg} a) + i \sin (\rho \operatorname{Arg} a)],$$

which we shall take as one of the values of a^{ρ}. In order to obtain all the values of the power a^{ρ} with an irrational index ρ, we shall ascribe all possible values to Arg a in the given expression.

Since two different values of ρ Arg a differ by a number of the form $2k\rho\pi$ which cannot be an integral multiple of 2π (k is an integer not equal to zero and ρ is an irrational number), therefore all values of a^{ρ}, corresponding to different values of Arg a, are all different.

Thus, we have defined the power a^{α} where α is an arbitrary real number. All values of the power are contained in the formula

$$a^{\alpha} = |a|_+^{\alpha} [\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)]. \tag{54}$$

We obtain a unique value of the power when α is an integer and different values, say q in number, when α is a rational number that can be represented by the irreducible fraction p/q. Finally, we obtain an infinite (denumerable) set of different values when α is an irrational number.

In order to define the concept of the power a^{α} for any complex index α, we note that the formula (54) may be written in the form

$$a^{\alpha} = e^{\alpha \ln |a|} [\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)] =$$
$$= \exp (\alpha \ln |a| + i \alpha \operatorname{Arg} a) = \exp (\alpha \operatorname{Ln} a).$$

The right-hand side of this formula has a meaning not only for real α but also for any complex α. Accordingly, we put, by definition,

$$a^{\alpha} = \exp (\alpha \operatorname{Ln} a) \tag{55}$$

for any complex α.

Evidently, when α is complex all values of a^{α} corresponding to different values of Ln a, or, what is the same thing, corresponding to different values of Arg a, are also different. In fact, two values of α Ln a differ by a number of the form $2\pi i\alpha$ which, when α is complex, cannot be an integral multiple of $2\pi i$.

We note that the power with an arbitrary index, generally speaking, obeys neither the rule of addition of indices on multiplication nor the rule of multiplication of indices when the power is further raised a power. Thus, if the number $\alpha_1(1-k) - \alpha_2 k$ is not an integer, for any integral value of k, we have

$$a^{\alpha_1} \cdot a^{\alpha_2} = \exp(\alpha_1 \operatorname{Ln} a) \cdot \exp(\alpha_2 \operatorname{Ln} a) = \exp(\alpha_1 \operatorname{Ln} a + \alpha_2 \operatorname{Ln} a) \neq$$
$$\neq \exp[(\alpha_1+\alpha_2) \operatorname{Ln} a] = a^{\alpha_1+\alpha_2},$$

and, similarly, if the number $\beta - k\alpha\beta$ is not an integer, for any integral value of k, we have

$$(a^{\alpha})^{\beta} = [\exp(\alpha \operatorname{Ln} a)]^{\beta} = \exp[\beta(\alpha \operatorname{Ln} a + 2k\pi i)] \neq$$
$$\neq \exp(\beta\alpha \operatorname{Ln} a) = a^{\alpha\beta}.$$

We now clarify, by examples, the definition of a power :

1) $1^{\sqrt{2}} = \exp(\sqrt{2} \operatorname{Ln} 1) = \exp(2k\pi i \sqrt{2}) = \cos(2k\pi\sqrt{2}) +$
$$+ i \sin(2k\pi\sqrt{2})$$
$$(k=0, \pm 1, \pm 2, \ldots);$$

2) $e^z = \exp(z \operatorname{Ln} e) = \exp[z(1+2k\pi i)] = \exp z \exp(2k\pi iz)$
$$(k=0, \pm 1, \pm 2, \ldots).$$

It is thereby clear that only one of the values of the power e^z coincides with $\exp z$. The other values are $\exp z \exp 2\pi iz$, $\exp z \exp(-2\pi iz)$, etc. In particular, one of the values of e^x (x being a real number) coincides with the real positive number $\exp x$. The other values are : $\exp x \exp 2\pi ix$, $\exp x \exp(-2\pi ix) \ldots$. Their number will be finite if x is rational and infinite if x is irrational.

Nevertheless, we shall use in our study the symbol e^z, customary in analysis, as congruent to $\exp z$. This use of a many-valued symbol is entirely analogous to the conventional use, in analysis, of the symbol $\sqrt[n]{a}$ (a being a real positive number) as the unique positive value of a radical.

3) $i^i = \exp(i \operatorname{Ln} i) = \exp\left[i\left(\frac{\pi}{2} i - 2k\pi i\right)\right] =$
$$= \exp\left[(4k-1)\frac{\pi}{2}\right] = e^{(4k-1)\frac{\pi}{2}}$$
$$(k=0, \pm 1, \pm 2, \ldots).$$

Thus, all the values of the power i^i are real positive numbers and among these we have numbers which are as large or as small as desired.

On the basis of the above definition of a power we can examine the following two many-valued functions :

$$z^\alpha \text{ and } a^z.$$

The first of these—a power with an arbitrary index—is defined quite generally, except for $z = 0$.

If α is a real integer, then z^α represents a rational function of a special kind. For $z = 0$ it is also defined when there is either a zero (if $\alpha>0$) or a pole (if $\alpha<0$). In case α is real rational nonintegral : $\alpha = p/q$ (q being a natural number and the fraction p/q being irreducible), z^α may be represented in the form

$$z^\alpha = \sqrt[q]{z^p}.$$

This is a many-valued, here a q-valued, function. For it, the points $z = 0$ and $z = \infty$ are branch points of order $q - 1$. In any domain G, obtained from the extended plane by drawing a Jordan curve connecting the branch points, we can isolate q different one-valued differentiable branches of the function. These branches continuously pass over to one another as the point z describes curves surrounding the origin of coordinates (or the point at infinity).

Finally, if α is not a real rational number (i.e. it is a real irrational or an arbitrary complex number), the function z^α would be infinitely many-valued ; all its values are expressed by the formula

$$z^\alpha = \exp(\alpha \operatorname{Ln} z).$$

For this function too the points $z = 0$ and $z = \infty$ are branch points. In this case, however, these branch points are of an infinite order.

In fact, as the point $z = 0$ is traversed once, say in the positive direction, the value of $\operatorname{Arg} z$ (changing continuously) increases by 2π and, consequently, the value of $\alpha \operatorname{Ln} z$ changes by $2\pi i\alpha$ and that of the function z^α acquires a factor $\exp(2\pi i\alpha) \neq 1$.

We now turn to the general exponential function a^z ($a\neq0$). For any finite value of z, it is defined by the formula

$$a^z = \exp(z \operatorname{Ln} a).$$

In order to get a specific one-valued branch, it is sufficient to fix one of the values $\operatorname{Ln} a = b$.

Assuming that one such value has been fixed, we obtain a one-valued, everywhere differentiable, function $\exp(bz)$. Taking all possible values of Ln a, we obtain all the possible one-valued branches of the function a^z. Since two values of Ln a differ by a term of the form $2k\pi i$, the two branches of the function a^z differ by a factor of the form $\exp(2k\pi iz)$, which also represents a one-valued, everywhere differentiable, function taking up the value 1 only for real integral values of z. However, the branches of the many-valued function will, in the case under examination, essentially differ in character from the branches of all the many-valued functions examined before. As a matter of fact, in all the previous examples there did exist such points in the extended plane (the so-called branch points), by traversing around which (along a closed Jordan curve) and making the values (of a certain branch) of the function change continuously, we could pass over continuously from one branch to another.

Here, however, such a possibility is excluded just because each branch represents a function that is one-valued and continuous over the whole of the finite plane. Whichever be the closed curve along which we traverse, we obtain on our return to the starting point the same number z (with, of course, a different value of the argument) and, consequently, the same value of the function $\exp(bz)$ [b being the fixed value of Ln a].

Thus, the many-valued function a^z has no branch point and its one-valued continuous branches cannot continuously pass over one another. This permits us to look upon these as independent functions (not connected with one another), being single-valued, everywhere differentiable, and hence entire functions :

$$\exp(z \ln a),\ \exp[z(\ln a + 2\pi i)],\ \exp[z(\ln a - 2\pi i)],\ \ldots\ .$$

The fact that all these different entire functions may be represented as branches of a single infinitely many-valued function a^z has no more significance than the fact that the functions $\sin z$ and $-\sin z$ may be looked upon as branches of the two-valued function $\sqrt{1-\cos^2 z}$ or that the hyperbolic functions $\sinh z$ and $\cosh z$ may be looked upon as two branches of the two-valued function $\frac{1}{2}\left[\exp z + \sqrt{\exp(-2z)}\right]$. (We draw the attention of the reader to the fact that the functions $\sqrt{1-\cos^2 z}$ and $\frac{1}{2}\left[\exp + z\sqrt{\exp(-2z)}\right]$, like the function a^z, do not possess branch points).

Fixing one of the branches of the function $z = a^w = \exp(bw)$ where b is one of the values of Ln a, we can examine the function which is

inverse to this branch. Evidently, we get

$$w = \frac{1}{b} \operatorname{Ln} z \quad (b = \ln a + 2k_0\pi i). \tag{56}$$

This function differs from Ln z only by a constant multiplier $1/b$. Since it follows from the relation (56) that

$$z = \exp(bw) = a^w \quad \text{(one of the values of } a^w\text{)},$$

w may be treated as the logarithm of z with base a.

Thus, we define the logarithm of an arbitrary complex number with a certain base a (a being a nonvanishing complex number) by the formula

$$\operatorname{Log}_a z = \frac{\operatorname{Ln} z}{\operatorname{Ln} a}, \tag{56'}$$

where one of the infinitely many values of Ln a (one and the same value b for all z) is fixed in the denominator.

This definition, therefore, requires not only that the base a of the system of logarithms be specified, but also that one of the values of Ln a be given.

We shall now explain what has been said above by a few examples.

1) $a = e$. If we fix the value of Ln e to be equal to 1, we would obtain

$$\operatorname{Log}_e z = \operatorname{Ln} z.$$

This is the usual definition of a natural logarithm. However, we might take the value of Ln e equal to, say, $1 + 2\pi i$. Then, we would have

$$\operatorname{Log}_e z = \frac{\operatorname{Ln} z}{1+2\pi i}.$$

The reader can easily verify that, according to this definition, of all real positive numbers only those of the form e^k (k being an integer) may have natural logarithms with unique real values.

2) $a = 10$. Taking the value of Ln 10 equal to $2.302585\ldots = \frac{1}{0.43429\ldots} = \frac{1}{M}$, we obtain

$$\operatorname{Log}_{10} z = M \operatorname{Ln} z = 0.43429\ldots \operatorname{Ln} z.$$

This definition of a common (with base 10) logarithm of arbitrary complex number z ($z \neq 0$) agrees with the usual definition of common (with base 10)

logarithms of real positive numbers. In fact, if $z = x > 0$, then taking the principal value of the logarithm, we have

$$\log x = 0.43429 \ldots \ln x.$$

3) $a = 1$. In this case, while defining the logarithm with base 1 by formula (56′), we must not use the principal value of Ln 1 as equal to zero. We take the value of Ln 1 equal to $2\pi i$. Then, by definition, we shall have

$$\operatorname{Log}_1 z = \frac{\operatorname{Ln} z}{2\pi i} = \operatorname{Arg} z - \frac{i}{2\pi} \ln |z|.$$

It follows, therefore, that all values of $\operatorname{Log}_1 z$ are real if $|z| = 1$ and complex if $|z| \neq 1$. Hence, only those numbers which are represented by the points of the unit circle have real logarithm with a base equal to unity. For these numbers the values of the logarithm coincide with the values of their arguments. Thus, $\operatorname{Arg} z$ for complex numbers with modulus 1 coincides with the logarithm of z when the base is equal to unity.

21. INVERSE TRIGONOMETRIC FUNCTIONS

In this section we shall discuss inverse trigonometric functions $\operatorname{Arccos} z$ and $\operatorname{Arctan} z$. The function $w = \operatorname{Arccos} z$ is defined by the equation

$$z = \cos w. \tag{57}$$

Replacing $\cos w$ by $\dfrac{\exp(iw) + \exp(-iw)}{2}$ and assuming for the sake of brevity $\exp(iw) = t$, we write (57) in the form

$$z = \frac{t + t^{-1}}{2}, \tag{58}$$

whence

$$t^2 - 2zt + 1 = 0 \tag{59}$$

and

$$t = z + \sqrt{z^2 - 1}. \tag{60}$$

Since both the roots of the quadratic equation (59) are nonvanishing (their product is equal to unity), the equation

$$\exp(iw) = t$$

does have roots (for an unknown w). One obtains

$$w = \frac{1}{i} \operatorname{Ln} t = \frac{1}{i} \operatorname{Ln} (z + \sqrt{z^2-1}). \tag{61}$$

(We note that all these calculations have already been performed in Art. 14 of this chapter).

Thus, the many-valued function $w = \operatorname{Arccos} z$ is expressed in terms of the logarithm and the quadratic root as

$$w = \operatorname{Arccos} z = \frac{1}{i} \operatorname{Ln} (z + \sqrt{z^2-1}). \tag{62}$$

First of all, the points $z = \pm 1$ are its branch points. In fact, as the point z traverses once any closed Jordan curve containing in its interior only one of these points, then one of the values of $\sqrt{z^2-1}$ is replaced by the other which merely differs from the former in sign. Here, the value $z + \sqrt{z^2-1} = t$, a root of the quadratic equation (59), is replaced by the other root of the same equation, viz. $1/t$ (we have already shown that the product of the two roots is equal to unity). Consequently, from the values $w = \frac{1}{i} \operatorname{Ln} t$ we pass over to the values $\frac{1}{i} \operatorname{Ln} \frac{1}{t}$, which actually differ from the former ones only when $t \neq 1/t$. But t is equal to $1/t$ only when $t = \pm 1$. As we see from (59) or (60), this is possible only when $z = \pm 1$. Since the Jordan curves which are traversed do not pass through the points $z = \pm 1$, hence the case mentioned above may not occur ; it may rather be affirmed that the values $w = \operatorname{Arccos} z$ get changed as a result of the traversals. Thus, the points ± 1 are branch points of the function Arccos z. These owe their existence to the presence of the square root in the formula (62). However, the formula (62) has the form $w = \frac{1}{i} \operatorname{Ln} t \; (t = z + \sqrt{z^2+1})$ and we can also expect branch points corresponding to the two branch points of Ln t, viz. $t = 0$ and $t = \infty$.

We know from Art. 14 of this chapter that to every single traversal by the point t of a circle, with its centre at the origin of coordinates, there corresponds a traversal by the point z of an ellipse, with its foci at ± 1 in the z-plane, and conversely.

Thus, to a single traversal by the point z of an ellipse with foci ± 1, there corresponds an alteration of Arg t by $\pm 2\pi$ and an alteration of $\frac{1}{i} \operatorname{Ln} t$ by $\pm 2\pi$. Since such an ellipse can belong to any previously given neighbourhood of the point $z = \infty$, this point is a branch point of Arccos z and is of an infinite order.

Of course, the traversal along any ellipse with foci at the points ± 1 can be looked upon as a traversal around the point $z = 0$. But none of these ellipses lies entirely in the neighbourhood $| z | \leqslant \rho$, where $\rho \leqslant 1$.

We shall show that neither the point $z = 0$ nor any point z_0 of the extended plane, excepting the points mentioned before ($z = \pm 1$ and $z = \infty$), can, in general, be a branch point of Arccos z. In fact, the formula (60) gives for $z = z_0$ two unequal values : t_0' and t_0'', other than 0, ± 1 and ∞ which satisfy the condition $t_0' t_0'' = 1$. We can take neighbourhoods U' and U'' of these points so small that they do not include the points 0 and ∞ and such that neither U' nor U'' contains the two points t_1, t_2 satisfying the condition $t_1 t_2 = 1$. In fact, if the point t_0' does not lie on the unit circle, for example, if $| t_0' | < 1$, then the point t_0'' also does not lie on this circle and $| t_0'' | > 1$. In this case, it is sufficient to take such neighbourhoods that one of them (U') lies entirely in the interior of the unit circle while the other (U'') lies entirely in its exterior (Fig. 32). If, however, t_0' lies

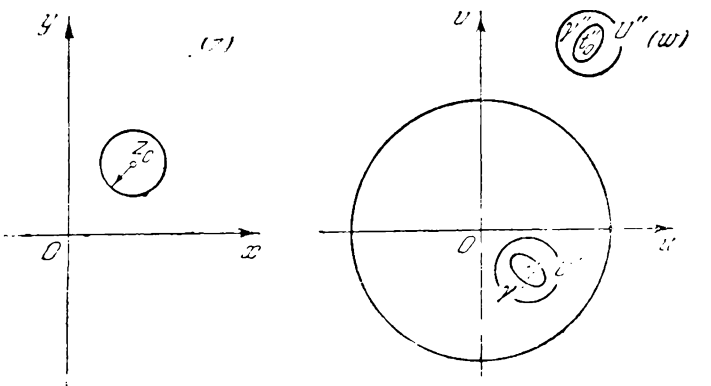


Fig. 32.

on the unit circle and is found, for instance, in the upper half-plane, then $t_0'' = 1/t_0'$ also lies on the unit circle and in the lower half-plane. In this case, it is sufficient to take one neighbourhood (U') in the upper half-plane and the other (U'') in the lower half-plane. In each of the neighbourhoods U' and U'' the function $z = \frac{1}{2}\left(t + \frac{1}{t}\right)$ will be single-sheeted (it takes up one and the same value only at the pair of points connected by the relation $t' \cdot t'' = 1$) and, hence, will map U' and U'' one-to-one onto certain domains g' and g'' in the z-plane, containing the point z_0 (image of the centres t_0' and t_0'' of the neighbourhoods U' and U'') in their interior.

We shall take the circles $| z - z_0 | = \rho$, with their centre at the point z_0, so small that they are contained in both the domains g' and g''.

Evidently, all neighbourhoods with sufficiently small radii satisfy this requirement. Then, by virtue of such a mapping (58) this circle corresponds to the closed Jordan curves γ' and γ'' in the neighbourhoods U' and U'' respectively. But none of these curves contains the point O in its interior. Hence, when z traverses the circle $|z-z_0| = \rho$, t traverses either the curve γ' [corresponding to one branch of the function (60)] or the curve γ'' [corresponding to the other branch of the function (60)]. If we fix the value of Arg t at any point γ' (or γ'') before the traversal, then this value (changing continuously) will, as a result of the traversal, return to the initial value (just because neither γ' nor γ'' contains the point $t = 0$ in its interior). Hence, the value of $\frac{1}{i}\operatorname{Ln} t = \operatorname{Arccos} z$ also returns to its initial value as a result of the traversal.

Thus, the point z_0, different from ± 1 and ∞, cannot be a branch point of Arccos z.

In order to obtain some domain in the z-plane such that in it we can isolate one-valued continuous branches of Arccos z, it is necessary to connect the branch points of this function by Jordan curves. We take, say, an infinite segment Δ of the real axis, connecting the points -1 and $+1$ through the point at infinity. This segment Δ is the boundary of a certain domain G.

From what we know of the function (58) it follows that it maps one-to-one the upper as well as the lower half-planes of t onto the domain G. The function $w = \frac{1}{i}\operatorname{Ln} t$ in turn maps each of these half-planes onto strips, in the w-plane, parallel to the imaginary axis and of width π; the upper half-plane, for example, onto the strip g_{2k-1}:

$$(2k-1)\pi < u < 2k\pi \quad (k=0, \pm 1, \pm 2, \ldots)$$

and the lower half-plane on to the strip g_{2k}:

$$2k\pi < u < (2k+1)\pi.$$

Thus, the said strips g_n are images of the domain G in the w-plane.

In order to fix a one-valued branch of Arccos z in the domain G it is sufficient to show to which of the strips g_n its values belong. Thus, we obtain the branches: $\operatorname{Arccos}_0 z$, $\operatorname{Arccos}_1 z$, $\operatorname{Arccos}_{-1} z, \ldots$. However, it is enough to fix the value of Arccos z at any one point of the domain G, say at the origin of coordinates. Then, the strip g_n in which this value falls represents a branch of Accros z.

The one-valued branches of Arccos z can, of course, be determined, in many other domains of the z-plane. Such domains are, for example, the

domain G' the boundary of which consists of the finite segment δ of the real axis connecting the points -1 and $+1$ and the positive part of the imaginary axis, or the domain G'' the boundary of which consists of the same said segment δ and the negative part of the imaginary axis.

We leave it to the reader to find out the domains, in the w-plane, onto which G' and G'' are mapped by the corresponding one-valued branches of the function Arccos z.

We now turn to examine the function $w = \text{Arctan } z$, inverse to the function $z = \tan w$. Expressing $\tan w$ in terms of $\sin w$ and $\cos w$, and further again in terms of the exponential functions, we obtain

$$z = \frac{1}{i}\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = \frac{1}{i}\frac{e^{2iw}-1}{e^{2iw}+1}, \tag{63}$$

or, assuming $e^{2iw} = \tau$:

$$z = \frac{1}{i}\frac{\tau-1}{\tau+1},$$

whence

$$\tau = \frac{1+iz}{1-iz},$$

and, finally,

$$w = \frac{1}{2i}\operatorname{Ln}\frac{1+iz}{1-iz}.$$

Thus, Arctan z is expressed by logarithm of a bilinear function

$$w = \text{Arctan } z = \frac{1}{2i}\operatorname{Ln}\frac{1+iz}{1-iz}. \tag{64}$$

We leave it to the reader to demonstrate that Arctan z has only two branch points, $\pm i$. The simplest domains in which we can isolate one-valued branches of Arctan z will be the domain D with the infinite segment Δ of the imaginary axis, connecting the branch points $-i$ and $+i$, as its boundary and the domain d with the finite segmemt δ connecting the same points as its boundary.

The reader can readily establish that the one-valued branches of Arctan z in the domain D map this domain one-to-one and conformally onto strips of width π, parallel to the imaginary axis:

$$k\pi - \frac{\pi}{2} < u < k\pi + \frac{\pi}{2} \qquad (k=0, \pm 1, \ldots),$$

and also that the one-valued branches of this function in the domain d map d onto similar strips: $k\pi < u < (k+1)\pi$.

4

COMPLEX TERM SERIES. POWER SERIES

1. CONVERGENT AND DIVERGENT SERIES

Let $\{w_n = u_n + iv_n\}$ be a sequence of complex numbers. The expression

$$w_1 + w_2 + w_3 + \ldots + w_n + \ldots, \tag{1}$$

or briefly $\sum_1^\infty w_n$, is called a *series*, the numbers $w_1, w_2, w_3, \ldots$ its *terms* while the sums $s_n = w_1 + w_2 + \ldots + w_n$ $(n = 1, 2, \ldots)$ are called *partial sums* of the series. If the sequence of the partial sums converges, then the series is said to be *convergent* and the limit of the sequence, $\lim_{n\to\infty} s_n = s$, is the *sum of the series*; in this case we write $\sum_1^\infty w_n = s$. If the sequence of the partial sums diverges, then the series is said to be *divergent*. It is obvious that these definitions include as special cases the usual definitions for the real term series. Observing that

$$s_n = \sum_1^n u_k + i \sum_1^n v_k$$

and that the relation $\lim_{n\to\infty} s_n = s$ is equivalent to the two relations: $\lim_{n\to\infty} \sum_1^n u_k = \operatorname{Re} s = \sigma$ and $\lim_{n\to\infty} \sum_1^n v_k = \operatorname{Im} s = \tau$, we find that *the convergence of a complex term series is equivalent to the simultaneous convergence of the two series composed, respectively, of the real and the imaginary parts,* $\sum_1^\infty u_n$ and $\sum_1^\infty v_n$, of the given series.

Applying the Cauchy criterion to the sequence $\{s_n\}$, we obtain the following *general criterion for the convergence of series*: the

series (1) converges if, and only if, there exists for any $\varepsilon > 0$ such $N(\varepsilon)$ that the inequalities $|s_{n+p}-s_n| < \varepsilon$ are satisfied for all $n > N(\varepsilon)$ and for any natural number p. As a special case ($p=1$), we obtain the *necessary condition for the convergence* of the series : $\lim\limits_{n\to\infty} w_{n+1} = 0$, i.e. $\lim\limits_{n\to\infty} w_n = 0$.

Series (1) is called *absolutely convergent* if the series consisting of the moduli of its terms, $\sum\limits_1^\infty |w_n|$, converges.

It follows from the general criterion of convergence that every absolutely convergent series converges. But the converse is not true, as will be evident from the well known example of the convergent but not absolutely convergent series $\sum\limits_1^\infty (-1)^{n-1}\frac{1}{n}$.

From the double inequality

$$\left.\begin{matrix}|u_n|\\|v_n|\end{matrix}\right\} \leqslant |w_n| \leqslant |u_n| + |v_n|,$$

it follows that the absolute convergence of the series $\sum\limits_1^\infty w_n$ is equivalent to the simultaneous convergence of the series $\sum\limits_1^\infty |u_n|$ and $\sum\limits_1^\infty |v_n|$, i.e. to the absolute convergence of the series $\sum\limits_1^\infty u_n$ and $\sum\limits_1^\infty v_n$. Consequently, the theorem that an *arbitrary change in the order of the terms does not affect the sum of the series* can be carried over to the absolutely convergent, complex term series. We can judge the absolute convergence of the series (1), i.e. the convergence of the series $\sum\limits_1^\infty |w_n|$, on the basis of any test for convergence of the series with positive terms, for instance, the Cauchy test (namely, *the series converges absolutely if,* $\sqrt[n]{|w_n|} \leqslant q < 1$ *for* $n \geqslant n_0$) or D'Alembert's test (namely, *the series converges absolutely, if* $\frac{|w_{n+1}|}{|w_n|} \leqslant q < 1$ *for* $n \geqslant n_0$).

Finally, we note that the sum or the difference of two convergent series $\sum\limits_1^\infty w_n' = s'$ and $\sum\limits_1^\infty w_n'' = s''$ can be expressed by the

convergent series

$$\sum_{1}^{\infty} (w_n' \pm w_n'') = s' \pm s''.$$

Given the additional condition for the absolute convergence of the given series, the series

$$\sum_{1}^{\infty} (w_1' w_n'' + w_2' w_{n-1}'' + \ldots + w_n' w_1'')$$

also converges absolutely and its sum is equal to $s' s''$. These propositions can be proved for complex term series exactly in the same way as the corresponding propositions for the real term series.

2. THE CAUCHY-HADAMARD THEOREM

The power series

$$a_0 + a_1 (z - z_0) + \ldots + a_n (z - z_0)^n + \ldots, \qquad (2)$$

where $a_0, a_1, \ldots, a_n, \ldots$ are given complex numbers and z is a complex variable, is the simplest example of a functional series, i.e. a series whose terms are certain functions of z. Such a series, generally speaking, converges for some values of z and diverges for others. This point is further elucidated by the following *Cauchy-Hadamard theorem.*

If $\overline{\lim}\ \sqrt[n]{|a_n|} = \Lambda$[1], *then for* $\Lambda = 0$ *the series* (2) *converges absolutely in the whole of the z-plane and for* $\Lambda = \infty$ *it converges only at the point* $z = z_0$ *and diverges for* $z \neq z_0$; *finally, for* $0 < \Lambda < \infty$, *it converges absolutely within the circle* $K: |z - z_0| < \frac{1}{\Lambda}$ *and diverges in its exterior.*

Proof. (*a*) First let $\Lambda = 0$. Then $\overline{\lim}\ \sqrt[n]{|a_n|} = \lim \sqrt[n]{|a_n|} = 0$ and, hence, for any z, $\lim \sqrt[n]{|a_n|\,|z - z_0|^n} = 0$. By virtue of the

[1]The number Λ $(-\infty \leqslant \Lambda \leqslant +\infty)$ is called the *upper limit* of the sequence of the real numbers $\{\alpha_n\}$: $\Lambda = \overline{\lim_{n\to\infty}}\ \alpha_n$, provided the following two conditions are satisfied:

1. For every $\Lambda' > \Lambda$ there exists an N such that $\alpha_n < \Lambda'$ for $n > N$.
2. There exists a subsequence $[\alpha_{k_n}]$ converging to Λ.

It has been proved in the courses on analysis (see, in this connection, G. M. Fichtenholtz, *A Course of Differential and Integral Calculus*, vol. 1, 3rd edition, 1951; Gostekhizdat, Section 42) that every sequence has either a finite or an infinite upper limit.

Cauchy test follows the convergence of the series $\sum_{0}^{\infty} |a_n| |z-z_0|^n$, i.e. the absolute convergence of the series (2).

(*b*) Let $\Lambda = \infty$. Then there exists an increasing sequence of indices $\{k_n\}$ such that $\sqrt[k_n]{|a_{k_n}|} \to \infty$. Hence, for any $z \neq z_0$, we have: $\sqrt[k_n]{|a_{k_n}||z-z_0|^{k_n}} = \sqrt[k_n]{|a_{k_n}|}\,|z-z_0| \to \infty$; this means that $|a_{k_n}||z-z_0|^{k_n} \to \infty$, i.e. the necessary condition for convergence is not satisfied by the series (2).

(*c*) Let $0 < \Lambda < \infty$. If $z = z_0$, then all the terms of the series (2), beginning from the second, vanish and hence the series converges absolutely. If $z \neq z_0$, and the point z lies in the interior of the circle K, then we can put $|z-z_0| = \frac{\theta'^2}{\Lambda}$, where $0 < \theta' < 1$. Since $\Lambda' = \frac{\Lambda}{\theta'} > \Lambda$, there exists, by the properties of the upper limit, an N_1 such that $\sqrt[n]{|a_n|} < \Lambda'$ for $n > N_1$. Then $\sqrt[n]{|a_n|\,|z-z_0|^n} < \Lambda' \cdot |z-z_0| = \frac{\Lambda}{\theta'}\frac{\theta'^2}{\Lambda} = \theta' < 1$ for $n > N_1$; from this, by the Cauchy test, follows the absolute convergence of the series (2).

Finally, if z lies outside the circle K, then $|z-z_0| = \frac{1}{\Lambda\theta}$ $(0<\theta<1)$.

By virtue of the properties of the upper limit there exists an increasing sequence of indices $\{k_n\}$ such that $\sqrt[k_n]{|a_{k_n}|} \to \Lambda$, and hence $\sqrt[k_n]{|a_{k_n}|\,|z-z_0|^{k_n}} \to \Lambda\,|z-z_0| = \frac{1}{\theta} > 1$.

Consequently, $|a_{k_n}|\,|z-z_0|^{k_n} \to \infty$, i.e. the necessary condition for convergence of (2) is not fulfilled.

The circle with centre at z_0 and of radius $1/\Lambda$, in the interior of which the power series converges absolutely and in the exterior of which it diverges, is called the *circle of convergence* of the power series and the number $R = 1/\Lambda$ the *radius of convergence*.

These definitions can also be extended to the extreme cases: $\Lambda = 0$ $(R = \infty)$ and $\Lambda = \infty$ $(R = 0)$. In the first of these cases, the circle of convergence is the whole finite complex plane and its exterior is an empty set. In the second case, the circle degenerates to the point z_0 and its exterior represents the whole of the plane with the point z_0 excluded.

For $0 < \Lambda < +\infty$, we still have the *circle of convergence* $|z-z_0| = R = 1/\Lambda$. The power series can behave differently (as regards its convergence properties) at the points of the circle.

We now examine, for instance, the series $\sum_{1}^{\infty} \frac{z^n}{n^\alpha}$ (α being a real

number). Here, $\Lambda = \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{1}{n^\alpha}} = \lim_{n\to\infty} n^{-\frac{\alpha}{n}} = 1$; hence, the radius of convergence is $R = 1$. On the circle $|z| = 1$ we have: $\left|\frac{z^n}{n^\alpha}\right| = \frac{1}{n^\alpha}$ and, hence, for $\alpha > 1$, the series converges absolutely at all the points of this circle and diverges for $\alpha \leqslant 0$. Now, let $0 < \alpha \leqslant 1$. Then, at the point $z = 1$ the series diverges. We shall show that it converges at all the other points of the unit circle. In fact, if $z = e^{i\theta}$ $(0<\theta<2\pi)$, then

$$\sum_{n+1}^{n+p} \frac{e^{ik\theta}}{k^\alpha} = e^{i(n+1)\theta}\left\{\sum_{j=0}^{p-2}\sigma_j\left[\frac{1}{(n+j+1)^\alpha}-\frac{1}{(n+j+2)^\alpha}\right]+\frac{\sigma_{p-1}}{(n+p)^\alpha}\right\},$$

where

$$\sigma_j = 1 + e^{i\theta} + \ldots + e^{ij\theta} = \frac{1-e^{i(j+1)\theta}}{1-e^{i\theta}} = e^{ij\frac{\theta}{2}}\frac{\sin\frac{j+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

Since $|\sigma_j| \leqslant \frac{1}{\sin\frac{\theta}{2}}$,

$$\left|\sum_{n+1}^{n+p}\frac{e^{ik\theta}}{k^\alpha}\right| < \frac{1}{\sin\frac{\theta}{2}}\left\{\sum_{j=0}^{p-2}\left[\frac{1}{(n+j+1)^\alpha}-\frac{1}{(n+j+2)^\alpha}\right]+\frac{1}{(n+p)^\alpha}\right\} =$$

$$= \frac{1}{\sin\frac{\theta}{2}(n+1)^\alpha} \to 0, \quad n \to \infty.$$

The convergence of the given series for $|z| = 1$ and $z \neq 1$ thereby follows.

Thus, a power series may either converge at all the points of its circle of convergence or diverge at all these points or, finally, converge at some of these points and diverge at others.

The following theorem follows as a simple corollary to the Cauchy-Hadamard theorem.

The first theorem of Abel. *If the series* $\sum_0^\infty a_n (z-z_0)^n$ *converges at a point* $z_1 \neq z_0$, *then it converges absolutely within the circle* $|z-z_0| < |z_1-z_0|$. As a matter of fact, it follows from the given condition that the point z_1 cannot lie in the exterior of the circle of convergence of the series. Hence, it lies either in its interior or on its boundary. In both these cases the circle $|z-z_0| < |z_1-z_0|$ belongs to the circle of convergence and, consequently, the series converges absolutely in it.

3. ANALYTICITY OF THE SUM OF A POWER SERIES

We shall now show that *within the circle of convergence* $|z-z_0| < R$ $(R>0)$ *of the power series* $\sum_0^\infty a_n (z-z_0)^n$ *its sum* $f(z)$ *is an analytic function; moreover, the derivative* $f'(z)$ *may be obtained by a termwise differentiation of the series:* $f'(z) = \sum_1^\infty n a_n (z-z_0)^{n-1}$. We shall find that the radius of convergence R' of the last series will also be equal to R. It is immediately obvious that this coincides with the radius of convergence of the series $\sum_0^\infty n a_n (z-z_0)^n$. But $\overline{\lim_{n\to\infty}} \sqrt[n]{n\,|a_n|} =$

$= \lim_{n\to\infty} n^{\frac{1}{n}} \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$, ; hence, $R' = (\overline{\lim_{n\to\infty}} \sqrt[n]{n\,|a_n|})^{-1} =$

$(\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|})^{-1} = R$.

Let z_1 be any point of the circle $|z-z_0| < R$. We take a point ζ such that $|z_1-z_0| < |\zeta-z_0| = \rho < R$.

Since the series $\sum_1^\infty n a_n (\zeta-z_0)^{n-1}$ converges absolutely, the inequality

$$\sum_{n+1}^{\infty} j\,|a_j|\,\rho^{j-1} < \varepsilon,$$

for any $\varepsilon > 0$, is satisfied when $j \geqslant n_0(\varepsilon) = n_0$. Putting $\sum_1^\infty n a_n (z-z_0)^{n-1} =$

$= \varphi(z)$, we shall get, for $z \neq z_1$, $|z-z_0| \leqslant \rho$:

$$\left|\frac{f(z)-f(z_1)}{z-z_1} - \varphi(z_1)\right| = \left|\sum_0^\infty a_n \frac{(z-z_0)^n-(z_1-z_0)^n}{z-z_1} - \sum_1^\infty n a_n (z_1-z_0)^{n-1}\right| =$$

$$= \left|\sum_1^\infty a_n [(z-z_0)^{n-1} + (z-z_0)^{n-2}(z_1-z_0) + \ldots + (z_1-z_0)^{n-1}] - \right.$$

$$\left. - \sum_1^\infty n a_n (z_1-z_0)^{n-1}\right| \leqslant \left|\sum_1^{n_0} a_n [(z-z_0)^{n-1} + \ldots \right.$$

$$\left. \ldots + (z_1-z_0)^{n-1} - n(z_1-z_0)^{n-1}]\right| + 2\sum_{n_0+1}^{\infty} n\,|a_n|\,\rho^{n-1}.$$

The first quantity tends to zero as $z \to z_1$ and, therefore, it will be less than ε for $|z-z_1| < \delta(\varepsilon)$, whereas the last one will be less than 2ε, owing to the nature of the choice of n_0.

Thus

$$\left|\frac{f(z)-f(z_1)}{z-z_1} - \varphi(z_1)\right| < 3\varepsilon \quad \text{for} \quad |z-z_1| < \delta(\varepsilon),$$

whence follows the validity of the theorem.

Applying this theorem to the sum of the series $\sum_1^\infty na_n(z-z_0)^{n-1} = -f'(z)$, we find that $f'(z)$ is also an analytic function within the circle $|z-z_0| < R$; moreover, $f''(z) = \sum_2^\infty n(n-1)a_n(z-z_0)^{n-2}$. Repeated application of this theorem leads us to the following result.

The sum of the power series $\sum_0^\infty a_n(z-z_0)^n = f(z)$ *is differentiable, without limit, within the circle of convergence:* $|z-z_0| < R\,(R>0)$. *The derivative of any order* p *is obtained by a termwise differentiation,* p *times, of the series. Thus*

$$f^{(p)}(z) = \sum_0^\infty n(n-1)\ldots(n-p+1)a_n(z-z_0)^{n-p} =$$

$$= \sum_{n=p}^\infty n(n-1)\ldots(n-p+1)a_n(z-z_0)^{n-p}.$$

Putting here $z = z_0$, we obtain

$$f^{(p)}(z_0) = p(p-1)\ldots 1 \cdot a_p,$$

whence

$$a_p = \frac{f^{(p)}(z_0)}{p!}.$$

This result is also true for $p = 0$, if we take $f^{(0)}(z) = f(z)$ and $0! = 1$. Hence, the power series may be rewritten in the form

$$f(z) = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

The last series is known as the *Taylor series* of the function $f(z)$. We have thus proved the theorem that *any power series with a positive radius of convergence is just a Taylor expansion for its sum.*

Let the series $\sum_0^\infty a_n (z-z_0)^n$ and $\sum_0^\infty b_n (z-z_0)^n$ have one and the same sum $f(z)$ in the neighbourhood $|z-z_0| < \rho$ ($\rho > 0$) of the point z_0. Then, as already proved, we shall have

$$a_n = b_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n=0, 1, 2, \ldots),$$

i. e. the corresponding coefficients of the two series are one-to-one equal. This proposition expresses the property of the *uniqueness of the expansion of a function into a power series.*

For illustrating the application of the above property we shall determine the form of the series in the powers of z, representing either even or odd functions of z.

Let the function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

be even, that is, $f(-z) = f(z)$. Then, the sum of the given series coincides with the sum of the series

$$f(-z) = a_0 - a_1 z + a_2 z^2 + \ldots + (-1)^n a_n z^n + \ldots,$$

whence, as proved above, we have

$$a_n = (-1)^n a_n \qquad (n=0, 1, 2, \ldots).$$

When n is odd, we have $a_n = -a_n$, whence $a_n = 0$ ($n=1, 3, 5, \ldots$) and, consequently,

$$f(z) = a_0 + a_2 z^2 + \ldots + a_{2m} z^{2m} + \ldots .$$

Similarly, we can show that when $f(z)$ is an odd function, that is, $f(-z) = -f(z)$, all the coefficients with even indices must vanish and, consequently,

$$f(z) = a_1 z + a_3 z^3 + \ldots + a_{2m-1} z^{2m-1} + \ldots .$$

We have proved above that the sum of any power series is an analytic function. Later, we shall prove that any analytic function can be expressed as a power series in certain neighbourhoods of its points of analyticity.

This enables us to extend the properties of the sum of a power series, established in this section, to all analytic functions.

4. UNIFORM CONVERGENCE

The concept of uniform convergence of series with real terms and their fundamental properties can be extended to series with complex terms also. Accordingly, the series $\sum_1^\infty f_j(z)$, where the function $f_j(z)$ is defined throughout the set E, is said to be *uniformly convergent in the set* $E_1 \subset E$ if it converges throughout E and, for any $\varepsilon > 0$, the inequality $\left| \sum_{n+1}^\infty f_j(z) \right| < \varepsilon$ is satisfied at all points of E_1, for $n > N(\varepsilon)$.

Putting $\sum_1^\infty f_j(z) = f(z)$, $\sum_1^\infty f_j(z) = S_n(z)$ and $\sum_{n+1}^\infty f_j(z) = f(z) - S_n(z) = R_n(z)$, we can express this condition in the form

$$\sup_{z \in E_1} |R_n(z)| \leqslant \varepsilon \text{ for } n > N(\varepsilon) \text{ or } \lim_{n\to\infty} \sup_{z \in E_1} |R_n(z)| = 0.$$

When the series converges in E_1, but not uniformly, we have: $\lim_{n\to\infty} R_n(z) = 0$, $z \in E_1$, but $\lim_{n\to\infty} \sup_{z \in E_1} |R_n(z)|$ either does not exist or is nonzero.

The condition of uniform convergence can also be expressed in the form $\left| \sum_{n+1}^{n+p} f_j(z) \right| < \varepsilon$ at all points of E_1, for $n \geqslant N(\varepsilon)$, and for any natural number p. Hence, we obtain the *test of uniform convergence (the so-called Weierstrass test)*: *if the series* $\sum_1^\infty u_n$, *with constant positive terms, converges and* $|f_n(z)| \leqslant u_n$, $z \in E_1$, $n \geqslant N_1$, *then the series* $\sum_1^\infty f_n(z)$ *converges uniformly (and absolutely) over* E_1. We shall now apply this test to the power series $\sum_0^\infty a_n(z-z_0)^n$, whose radius of convergence is $R > 0$; we shall show that for $r < R$ it converges uniformly in any circle $|z - z_0| \leqslant r$.

Let z_1 be a point of the circle of convergence, lying outside the circle $|z-z_0| \leqslant r$. Then, $r < |z_1-z_0| = \rho < R$ and, consequently, the

inequality $| a_n (z-z_0)^n | \leqslant | a_n (z_1-z_0)^n |$ is satisfied at every point of the circle $| z-z_0 | \leqslant r$. But the series with constant terms, $| a_n (z_1-z_0)^n |$, converges ; hence, the *series* $\sum_0^\infty a_n (z-z_0)^n$ *converges uniformly in the circle* $| z - z_0 | \leqslant r$ $(r < R)$.

We shall now show that there exist power series which converge non-uniformly in the circle of convergence. The simplest example of such a series is the geometric series $\sum_0^\infty z^n$. Here, $R = 1$, $S_n(z) = 1 + z + \ldots + z^n = \frac{1-z^{n+1}}{1-z}$, $f(z) = \frac{1}{1-z}$, $R_n(z) = \frac{z^{n+1}}{1-z}$ and $\sup_{|z|<1} | R_n(z) | = \sup_{|z|<1} \frac{|z|^{n+1}}{|1-z|} = +\infty$ for any n, i.e. the condition for uniform convergence is not satisfied.

On the other hand, there exist power series which converge uniformly in a closed circle of convergence. An example of this is the series $\sum_1^\infty \frac{z^n}{n^2}$. Here, $R = 1$ and $\left|\frac{z^n}{n^2}\right| \leqslant \frac{1}{n^2}$ for $z \leqslant 1$. With the help of the Weierstrass test, we find that the series converges uniformly in the closed unit circle.

We note here the formulation and the proof of the well known theorem, viz. *if the functions* $f_i(z)$ *are continuous in* E *and the series* $\sum_1^\infty f_i(z)$ *converges uniformly in* E, *then the sum of the series* $f(z)$ *is also continuous in* E, which can, without any change, be extended to the uniformly convergent complex term series.

5

INTEGRATION OF A FUNCTION OF A COMPLEX VARIABLE

1. INTEGRAL OF A FUNCTION OF A COMPLEX VARIABLE

Let L be a rectifiable curve in the complex plane and let $z = \lambda(t) = x(t) + iy(t)$ $(\alpha \leqslant t \leqslant \beta)$ be its equation.[1] We can choose one of the two possible directions on L; the first corresponds to the increase and the second to the decrease of the parameter t. Having chosen one of them arbitrarily, we shall hereafter understand by the symbol L the given curve with a fixed direction and by $-L$ the curve in the opposite direction. Suppose, for the sake of definiteness, that the direction chosen corresponds to the increasing parameter t. Then, we shall call the point $z_0 = \lambda(\alpha)$ the initial and the point $z' = \lambda(\beta)$ the terminal point of L. To each system of values of t: $t_0 = \alpha < t_1 < t_2 < \ldots < t_n = \beta$ there corresponds a division T of the curve L into arcs $l_0, l_1, \ldots, l_{n-1}$. Then, $z_k = \lambda(t_k) = x_k + iy_k$ is the initial point of the arc l_k and $z_{k+1} = \lambda(t_{k+1})$ $(z_n = z')$ is its terminal point. If $w = f(z) = u(x, y) + iv(x, y)$ is a function which is one-valued and continuous on L then, choosing in each segment $t_k \leqslant t \leqslant t_{k+1}$ one value of the parameter $t = \tau_k$, we obtain on each of the arcs l_k a point $\zeta_k = \lambda(\tau_k)$ at which we shall take the value of the function $f(z)$ for the purpose of constructing the sum $\sum_0^{n-1} f(\zeta_k)(z_{k+1} - z_k)$, corresponding to the given division T. Assuming, for the sake of brevity, $f(\zeta_k) = u(\xi_k, \eta_k) + iv(\xi_k, \eta_k) = u_k + iv_k$, $z_{k+1} - z_k = (x_{k+1} - x_k) + i(y_{k+1} - y_k) = \Delta x_k + i\Delta y_k$, we obtain

$$\sum_0^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \sum_0^{n-1} (u_k \Delta x_k - v_k \Delta y_k) + i \sum_0^{n-1} (v_k \Delta x_k + u_k \Delta y_k).$$

[1] For definition of a rectifiable curve, see G. M. Fichtenholz, *Foundations of Mathematical Analysis*, vol. 1, p. 199, Gostekhizdat, 1955.

It is obvious that the real and imaginary parts of the foregoing sum are themselves sums for two real functions of x and y which are introduced into analysis while defining the curvilinear integral of the form $\int_L P(x, y)\,dx + Q(x, y)\,dy$. The first sum is constructed from the functions $u(x, y)$ and $-v(x, y)$ and for the given division T of the curve L, while the second is constructed from $v(x, y)$ and $u(x, y)$ and for the same division. Since each of the functions of both the pairs is continuous on L and further the curve itself is rectifiable, therefore, by a well known theorem of analysis[1] these sums tend to finite limits $\int_L u(x, y)\,dx - v(x, y)\,dy$ and $\int_L v(x, y)\,dx + u(x, y)\,dy$ for any sequence of the divisions T satisfying the condition $\delta = \max(t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1}) \to 0$. These limits, as we know, do not depend on a given sequence of divisions T but on the choice of the points $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ on the arcs of each division. Thus, there exists the limit

$$\lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) =$$

$$= \int_L u(x, y)\,dx - v(x, y)\,dy + i \int_L v(x, y)\,dx + u(x, y)\,dy.$$

This limiting sum is called the *integral of the function* $f(z)$ *along the curve* L (in a given direction) and is denoted by the symbol $\int_L f(z)\,dz$:

$$\lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \int_L f(z)\,dz.$$

L is correspondingly referred to as *the path or the contour of integration.*

It follows from the definition that the integrals of $f(z)$ along L and $-L$ are connected by the relation

$$\int_{-L} f(z)\,dz = -\int_L f(z)\,dz.$$

[1] See, in this connection, G. M. Fichtenholz, *A Course of Differential and Integral Calculus*, vol. 3, p. 560, Gostekhizdat, 1949.

Examples: *a*) Let $f(z) = 1$, then the sum

$$\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1}-z_k) = \sum_{k=0}^{n-1} (z_{k+1}-z_k) = z_n - z_0 = z' - z_0,$$

and, hence, $\int_L dz = z' - z_0$. In particular, when L is a closed curve, i. e. $z' = z_0$: $\int_L dz = 0$.

b) Let $f(z) = z$. Then, putting $\zeta_k = z_k$ $(k=0, 1, 2, \ldots, n-1)$, we obtain the sum $\sum_{k=0}^{n-1} z_k(z_{k+1}-z_k)$. If we put $\zeta_k = z_{k+1}$ $(k=0, 1, 2, \ldots, n-1)$, then the sum assumes the form $\sum_{k=0}^{n-1} z_{k+1}(z_{k+1}-z_k)$. Since the two sums have a common limit, their arithmetic mean must also have the same limit $\int_L z\,dz$:

$$\int_L z\,dz = \lim_{\delta\to 0} \frac{1}{2} \sum_{k=0}^{n-1} (z_{k+1}+z_k)(z_{k+1}-z_k) = \lim_{\delta\to 0} \frac{1}{2} \sum_{k=0}^{n-1} (z_{k+1}^2 - z_k^2) =$$

$$= \lim_{\delta\to 0} \frac{1}{2} (z_n^2 - z_0^2) = \frac{1}{2} (z'^2 - z_0^2).$$

If L is a closed curve, then we again obtain: $\int_L z\,dz = 0$.

c) Next, we calculate the integral $\int_{|z-a|=\rho} \frac{dz}{z-a}$, where the circle $|z-a| = \rho$ is traversed in the anti-clockwise direction. The equation of the circle may be written in the form $z = a + \rho e^{it}$ $(0 \leqslant t \leqslant 2\pi)$. We choose a division $t_k = \frac{2k\pi}{n}$ and put $\tau_k = \frac{t_k + t_{k+1}}{2} = \frac{2k+1}{n}\pi$ $(k=0, 1, 2 \ldots, n-1)$; then, $z_k = a + \rho e^{\frac{2k\pi i}{n}}$ and $\zeta_k = a + \rho e^{\frac{(2k+1)\pi i}{n}}$. The corresponding sum takes the form

$$\sum_0^{n-1} \frac{1}{\rho e^{(2k+1)\frac{\pi i}{n}}} \rho \left(e^{\frac{2(k+1)\pi i}{n}} - e^{\frac{2k\pi i}{n}} \right) =$$

$$= \sum_0^{n-1} \left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}} \right) = n \left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}} \right) = n2i \sin \frac{\pi}{n}.$$

Hence

$$\int\limits_{|z-a|=\rho} \frac{dz}{z-a} = \lim_{n\to\infty} 2ni \cdot \sin\frac{\pi}{n} = 2\pi i .$$

2. THE PROPERTIES OF INTEGRALS

The following properties of the integral $\int\limits_L f(z)\,dz$ follow as corollaries from the corresponding well known properties of curvilinear integrals of the type $\int\limits_L P(x, y)\,dx + Q(x, y)\,dy$. These are

I. $$\int\limits_L \sum_{k=1}^{p} c_k f_k(z) = \sum_{k=1}^{p} c_k \int\limits_L f_k(z)\,dz ;$$

II. $$\int\limits_{L_1+L_2+\ldots+L_q} f(z)\,dz = \sum_{j=1}^{q} \int\limits_{L_j} f(z)\,dz.$$

Here, $L_1 + L_2 + \ldots + L_q$ denotes a curve composed of the arcs L_j, such that the terminal point of L_j coincides with the initial point of L_{j+1} ($j = 1, 2, \ldots, q-1$).

Very often we are required to estimate the modulus of the integral from above. We have, for the modulus of the corresponding sum, the estimate

$$\left|\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1}-z_k)\right| \leqslant \sum_{k=0}^{n-1} |f(\zeta_k)|\,|z_{k+1}-z_k|.$$

It is obvious that $|z_{k+1}-z_k|$ is the length of the chord connecting the points z_k and z_{k+1}; hence, $|z_{k+1}-z_k| \leqslant$ length of $l_k = \sigma_k$. Consequently,

$$\left|\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1}-z_k)\right| \leqslant \sum_{k=0}^{n-1} |f(\zeta_k)|\,\sigma_k.$$

Now, passing over to the limit, under the condition that $\delta = \max(t_1-t_0, \ldots, t_n-t_{n-1}) \to 0$, we obtain

III. $$\left|\int\limits_L f(z)\,dz\right| \leqslant \int\limits_L |f(z)|\,d\sigma.$$

On the right-hand side here we have a curvilinear integral of the form $\int_L f(\sigma)\, d\sigma$ where σ is the length of arc, measured from the initial point of L up to any arbitrary point thereof (along the direction chosen). If $|f(z)| \leqslant M$ $(z \in L)$, then from the estimate obtained above we derive a simpler but, in general, less precise estimate :

$$\text{IV.} \quad \left| \int_L f(z)\, dz \right| \leqslant M \cdot \text{length } L.$$

In particular,

$$\text{IV}'. \quad \left| \int_L f(z)\, dz \right| \leqslant \sup_L |f(z)| \cdot \text{length } L.$$

We also note the following formula for the *termwise integration of a series* :

$$\text{V.} \quad \int_L \sum_1^\infty f_n(z)\, dz = \sum_1^\infty \int_L f_n(z)\, dz.$$

This formula is valid, for instance, under the following conditions ; viz. the functions $f_n(z)$ $(n = 1, 2, 3, \ldots)$ are continuous on L and the series $\sum_1^\infty f_n(z)$ converges uniformly over L. The proof is exactly the same as in the case of functions of a real variable. Examples from general courses of analysis show (in the particular case when L is a segment of the real axis and all the functions $f_n(z) = f_n(x)$ take up real values) that the mere convergence of the series is insufficient for the formula V to be valid.

3. REDUCTION TO THE CALCULATION OF AN ORDINARY INTEGRAL

Let L be a smooth curve ; this means that we can have for it a parametric representation

$$z = \lambda(t) = x(t) + iy(t) \quad (\alpha \leqslant t \leqslant \beta)$$

such that $\lambda(t)$ has a continuous nonvanishing derivative $\lambda'(t) = x'(t) + iy'(t)$ on the segment $[\alpha, \beta]$. A geometrically smooth curve is characterized by the fact that at every point it has a tangent whose angle of inclination to the real axis [which is equal to $\operatorname{Arg} \lambda'(t)$] changes continuously as the point of contact moves along the curve.

For the integral $\int_L P(x, y)\,dx + Q(x, y)\,dy$, taken along a smooth curve, we find during our study of analysis the formula

$$\int_L P(x, y)\,dx + Q(x, y)\,dy =$$

$$= \int_{\alpha}^{\beta} \{P[x(t), y(t)]\,x'(t) + Q[x(t), y(t)]\,y'(t)\}\,dt,$$

which reduces the calculation of a curvilinear integral to that of an ordinary one. Hence,

$$\int_L f(z)\,dz = \int_L u\,dx - v\,dy + i\int_L v\,dx + u\,dy =$$

$$= \int_{\alpha}^{\beta} \{u[x(t), y(t)]\,x'(t) - v[x(t), y(t)]\,y'(t)\}\,dt +$$

$$+ i\int_{\alpha}^{\beta} \{v[x(t), y(t)]\,x'(t) + u[x(t), y(t)]\,y'(t)\}\,dt.$$

We note that the ordinary integral $\int_{\alpha}^{\beta} f(t)\,dt$ can be regarded as a special case of the complex integral $\int_L f(z)\,dz$ defined in Art. 1. Here, L represents a segment of the real axis traversed in the direction from α to β, while $f(z)$ takes up real values. Hence, we can apply property I of Art. 2 to the integral on the right-hand side of the last equation. Thus, we obtain

$$\int_L f(z)\,dz = \int_{\alpha}^{\beta} \{u[x(t), y(t)]\,x'(t) - v[x(t), y(t)]\,y'(t) +$$

$$+ iv[x(t), y(t)]\,x'(t) + iu[x(t), y\,t)]\,y'(t)\}\,dt =$$

$$= \int_{\alpha}^{\beta} \{u[x(t), y(t)] + iv[x(t), y(t)]\}\,[x'(t) + iy'(t)]\,dt.$$

Obviously, $u[x(t), y(t)] + iv[x(t), y(t)] = f[\lambda(t)]$ and $x'(t) + iy'(t) = \lambda'(t)$; hence, we finally obtain

$$\int_L f(z)\,dz = \int_\alpha^\beta f[\lambda(t)]\,\lambda'(t)\,dt.$$

This formula reduces the calculation of the complex integral to that of an ordinary integral (of a complex function).

Example. We examine the integral $\int_{|z-a|=\rho} \frac{dz}{z-a}$ (the circle is traversed in the anti-clockwise direction). Here, the equation of the curve of integration is $z = a + \rho e^{it}$ $(0 \leqslant t \leqslant 2\pi)$ and $\lambda'(t) = \frac{dz}{dt} = i\rho e^{it}$. Hence,

$$\int_{|z-a|=\rho} \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{\rho e^{it}}\, i\rho e^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Of course, this result agrees with that of Art. 1.

4. THE INTEGRAL THEOREM OF CAUCHY

Now, we proceed to prove a theorem which occupies a key position in the theory of analytic functions of a complex variable.

The integral theorem of Cauchy. *If G be a simply-connected domain of the finite plane and $f(z)$ be a one-valued, analytic function in this domain, then for any closed rectifiable curve lying in this domain the integral of $f(z)$ along L will be equal to zero*:

$$\int_L f(z)\,dz = 0.$$

We see that this theorem may be easily derived from the D'Alembert-Euler equations and the Green formula, under the additional assumption that the derivative $f'(z)$ is continuous in G. In fact, the Green formula,

$$\int_L P\,dx + Q\,dy = \int_D\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where D is the interior of a closed rectifiable Jordan curve L, is usually derived under the following assumptions:

a) any straight line, parallel to a coordinate axis, intersects L at not more than two points (excluding only two permitted extreme situations, viz. in the direction of each axis, for which intersection through a rectilinear segment is possible) ;

b) P, Q, $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are continuous in the closed domain D. Considering that L satisfies the condition *a*) and noting that the continuity of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ (and, of course, of the functions u and v themselves) follows from the continuity of $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, we obtain

$$\int_L f(z)\,dz = \int_L (u\,dx - v\,dy) + i\int_L (v\,dx + u\,dy) =$$

$$= \int_D\int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\int_D\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$$

The expressions under the signs of double integrals reduce to zero because of the D'Alembert-Euler equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y};$$

hence,

$$\int_L f(z)\,dz = 0.$$

Later, we shall find that the derivative of an analytic function is always analytic and, hence, continuous. However, this inference rests on the integral theorem of Cauchy. To avoid 'circular argument', we give a proper proof of this theorem without assuming the continuity of $f'(z)$. Such a proof was first given by E. Goursat and was later simplified by A. Pringsheim.

We shall first give a lemma before giving the proof itself. According to this, the general case of a rectifiable curve will get reduced to a very simple case where L is a triangular contour.

Lemma. *If the function $f(z)$ is continuous in a simply-connected domain G and over any triangular contour Δ contained in G: $\int_\Delta f(z)\,dz = 0$,*

then $\int_L f(z)\,dz = 0$ for any closed curve L contained in G.

Proof. First let L be a closed polygon P. If the number of its vertices is $n > 3$ and it has no self-intersections, then there exists a diagonal MN, which lies in the interior of the polygon (excluding the points M and N lying on P) and divides the interior of P into two polygons $MNAM$ and $NMBN$ (Fig. 33), each of these having a number of sides less than n (prove this !). Since

$$\int_{MNAM} f(z)\,dz = \int_{MN} f(z)\,dz + \int_{NAM} f(z)\,dz$$

and

$$\int_{NMBN} f(z)\,dz = \int_{NM} f(z)\,dz + \int_{MBN} f(z)\,dz,$$

we have

$$\int_{MNAM} f(z)\,dz + \int_{NMBN} f(z)\,dz = \int_{NAM} f(z)\,dz + \int_{MBN} f(z)\,dz = \int_{P} f(z)\,dz.$$

If the lemma is true for a polygon with the number of vertices less than n, then it follows that it will also be true for a polygon with n vertices. But, by the assumption, this holds for triangles. Consequently, by induction, it also holds for any polygon without self-intersections.

We further examine a polygon P with n vertices, having self-intersections (Fig. 34). We shall traverse it in a given direction, starting from

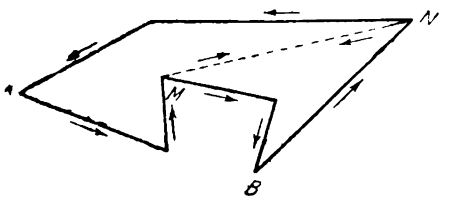


Fig. 33.

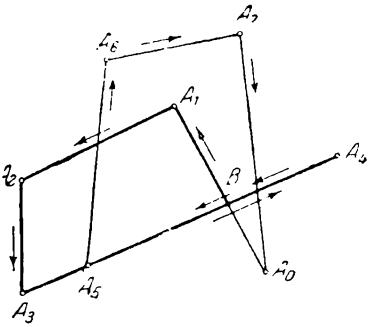


Fig. 34.

the point A_0 until some side, for the first time, meets one of the sides already transversed. In our example, this side is A_3A_4, which intersects the side A_0A_1 at the point B. Then, the closed polygon $BA_1A_2A_3B$ will not have any self-intersection ; consequently, as proved,

$$\int_{P} f(z)\,dz = \int_{BA_1A_2A_3B} f(z)\,dz + \int_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz = \int_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz.$$

The number of vertices of the new polygon $A_0BA_4A_5A_6A_7$ is less than n because we have introduced here only one new (in comparison with P) vertex B and left out, along with $BA_1A_2A_3B$, at least two vertices of the original polygon. It is, however, necessary here to note one possibility which has not yet been examined above. We shall illustrate this with reference to the polygon $A_0BA_4A_5A_6A_7$. If we traverse it, starting from A_0, in a direction determined by the above-mentioned order of vertices then, on our path from A_4 to A_5, we shall come across points of the side BA_4 which had already been traversed. In this case we should treat separately the two-sided figure BA_4B, the integral along which is evidently equal to zero; we thus obtain:

$$\int\limits_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz = \int\limits_{BA_4B} f(z)\,dz + \int\limits_{A_0BA_5A_6A_7A_0} f(z)\,dz = \int\limits_{A_0BA_5A_6A_7A_0} f(z)\,dz.$$

The number of vertices of the last polygon is less by unity than the number of vertices of the previous one because the vertex A_4 is left out and now some other one has been introduced in its place.

Thus, in all cases a polygon with self-intersections can, without any change in the value of the integral, be replaced by some other polygon with a smaller number of sides. Consequently, it is also possible here to complete the proof by induction, starting with the triangle (for a 'biangle' also, which is represented by a rectilinear segment traversed twice in opposite directions, this assumption is, evidently, valid).

Finally, let L be any arbitrary closed rectifiable curve belonging to G. We shall examine here any bounded closed sub-domain E of the domain G, containing L. To be definite, let E be the set of all those points whose distances from L do not exceed some $\delta > 0$ (δ must be less than all the distances between L and the boundary of G).

The function $f(z)$, continuous on E, is uniformly continuous on it and, for any $\varepsilon > 0$, we can find a $\rho > 0$ such that $|f(\zeta_1) - f(\zeta_2)| < \varepsilon$ if $|\zeta_1 - \zeta_2| < \rho$ and $\zeta_j \in E$ ($j = 1, 2$). We now fix on L, in a definite direction (the direction of integration), the points $z_0, z_1, \ldots, z_n = z_0$, dividing it into arcs $l_0, l_1, \ldots, l_{n-1}$ such that $\max\limits_{j=0,1,\ldots,n-1}$ (length l_j) $< \min(\delta, \rho)$. Connecting each pair of points z_j and z_{j+1} ($j = 0, 1, \ldots, n-1$) by a straight line segment p_j, we obtain the chords of the curve L. Since length $p_j \leqslant$ length l_j, all p_j are contained in the set E. The totality of all these chords constitutes the closed broken line P, inscribed in L and

lying in E. Since $\int_P f(z)\,dz = 0$, as proved above, we have

$$\left|\int_L f(z)\,dz\right| = \left|\int_L f(z)\,dz - \int_P f(z)\,dz\right| =$$

$$= \left|\sum_{j=0}^{n-1}\int_{l_j} f(z)\,dz - \sum_{j=0}^{n-1}\int_{p_j} f(z)\,dz\right| \leqslant \sum_{j=0}^{n-1}\left|\int_{l_j} f(z)\,dz - \int_{p_j} f(z)\,dz\right|.$$

Noting that

$$\int_{l_j} f(z_j)\,dz = f(z_j)\,(z_{j+1}-z_j) = \int_{p_j} f(z_j)\,dz,$$

we obtain

$$\left|\int_{l_j} f(z)\,dz - \int_{p_j} f(z)\,dz\right| \leqslant \left|\int_{l_j}[f(z)-f(z_j)]\,dz + \int_{p_j}[f(z) \quad f(z_j)]\,dz\right| \leqslant$$

$$\leqslant \sup_{z\in l_j}|f(z)-f(z_j)|\ \text{length}\ l_j + \sup_{z\in p_j}|f(z)-f(z_j)|\ \text{length}\ p_j.$$

But any two points of the arc l_j, or of the chord p_j, are separated from one another by a distance less than ρ; hence,

$$\sup_{z\in l_j}|f(z)-f(z_j)| < \varepsilon, \quad \sup_{z\in p_j}|f(z)-f(z_j)| < \varepsilon$$

and

$$\left|\int_{l_j} f(z)\,dz - \int_{p_j} f(z)\,dz\right| < \varepsilon\,(\text{length}\ l_j + \text{length}\ p_j) \leqslant 2\varepsilon\ \text{length}\ l_j.$$

Consequently

$$\left|\int_L f(z)\,dz\right| \leqslant 2\varepsilon \sum_{j=0}^{n-1}\text{length}\ l_j = 2\varepsilon\cdot\text{length}\ L,$$

whence, because of the fact that ε is arbitrarily small, $\int_L f(z)\,dz = 0$.

5. PROOF (continued)

Thus, for establishing the integral theorem of Cauchy it is sufficient to prove it for the case when L is the contour Δ of any triangle lying in the domain G. Putting

$$\left|\int_\Delta f(z)\,dz\right| = M \geqslant 0,$$

we have to prove that $M = 0$. We divide Δ into four triangles Δ^{I}, Δ^{II}, Δ^{III} and Δ^{IV}, by rectilinear segments connecting the middle points of the three sides of Δ (Fig. 35). Since

$$\int_{\Delta^{I}} = \int_{NAM} + \int_{MN}, \quad \int_{\Delta^{II}} = \int_{MBP} + \int_{PM}, \quad \int_{\Delta^{III}} = \int_{PCN} + \int_{NP},$$

$$\int_{\Delta^{IV}} = \int_{NM} + \int_{MP} + \int_{PN},$$

we have

$$\int_{\Delta^{I}} + \int_{\Delta^{II}} + \int_{\Delta^{III}} + \int_{\Delta^{IV}} = \int_{NAM} + \int_{MBP} + \int_{PCN} = \int_{\Delta},$$

and, hence,

$$M = \left| \int \right| \leqslant \left| \int_{\Delta^{I}} \right| + \left| \int_{\Delta^{II}} \right| + \left| \int_{\Delta^{III}} \right| + \left| \int_{\Delta^{IV}} \right|;$$

consequently, at least one of the moduli $\left| \int_{\Delta^{I}} \right|$, $\left| \int_{\Delta^{II}} \right|$, $\left| \int_{\Delta^{III}} \right|$ and $\left| \int_{\Delta^{IV}} \right|$ is not less than $\frac{M}{4}$. We now denote by Δ_1 such of the triangles Δ^{I}, Δ^{II}, Δ^{III} and Δ^{IV} for which

$$\left| \int_{\Delta_1} f(z)\, dz \right| \geqslant \left| \frac{M}{4} \right|.$$

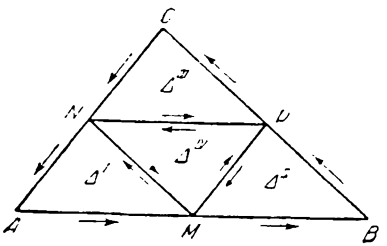


Fig. 35.

Repeating the same procedure for Δ_1 as has been applied above for Δ, we obtain a triangle Δ_2 (one of the four triangles into which the triangle Δ_1 is divided by rectilinear segments connecting the middle points of its sides) such that

$$\left| \int_{\Delta_2} f(z)\, dz \right| \geqslant \frac{M}{4^2}.$$

In this way we obtain a sequence of triangles $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$, each of which lies within the preceding one (we denote by the same symbol Δ_n both the contour of the triangle and the closed domain bounded by it; the meaning of Δ_n will be clear from the context). Thus

$$\left| \int_{\Delta_n} f(z)\, dz \right| \geqslant \frac{M}{4^n}$$

and

$$\text{length } \Delta_n = \frac{\text{length } \Delta_{n-1}}{2} = \frac{\text{length } \Delta}{2^n}.$$

We now make use of the analyticity of $f(z)$. All the triangles Δ_j ($\Delta_0 \equiv \Delta$) have a common point ζ belonging to the closed domain Δ and, consequently, to the domain G. Since $f(z)$ is differentiable at this point, there exists for any $\varepsilon > 0$, $\delta > 0$ such that

$$\left| \frac{f(z)-f(\zeta)}{z-\zeta} - f'(\zeta) \right| < \varepsilon \quad \text{for} \quad |z-\zeta| < \delta.$$

Putting

$$\alpha(z, \zeta) = \frac{f(z)-f(\zeta)}{z-\zeta} - f'(\zeta),$$

we get

$$f(z) = f(\zeta) + f'(\zeta)(z-\zeta) + \alpha(z, \zeta)(z-\zeta),$$

where $|\alpha(z, \zeta)| < \varepsilon$ for $|z-\zeta| < \delta$.

If n is sufficiently large ($n \geqslant n_0$), the triangle Δ_n containing ζ will be contained in the circle $|z-\zeta| < \delta$; hence, as $\int_{\Delta_n} dz = \int_{\Delta_n} z\,dz = 0$ (*cf.* Art. 1), we obtain

$$\left| \int_{\Delta_n} f(z)\,dz \right| = \left| \int_{\Delta_n} [f(\zeta) + f'(\zeta)(z-\zeta) + \alpha(z, \zeta)(z-\zeta)]\,dz \right| =$$

$$= \left| f(\zeta) \int_{\Delta_n} dz + f'(\zeta) \int_{\Delta_n} z\,dz - \zeta f'(\zeta) \int_{\Delta_n} dz + \int_{\Delta_n} \alpha(z, \zeta)(z-\zeta)\,dz \right| <$$

$$< \varepsilon \sup_{z \in \Delta_n} |z-\zeta| \cdot \text{length } \Delta_n < \varepsilon (\text{length } \Delta_n)^2 = \varepsilon \frac{(\text{length } \Delta)^2}{4^n}.$$

Comparing the estimates obtained from below or from above of $\left| \int_{\Delta_n} f(z)\,dz \right|$, we derive:

$$\frac{M}{4^n} \leqslant \varepsilon \frac{(\text{length } \Delta)^2}{4^n};$$

therefore,

$$M \leqslant \varepsilon (\text{length } \Delta)^2.$$

But ε is arbitrarily small; hence, $M = 0$. Thus, the theorem is proved.

6. APPLICATION TO THE CALCULATION OF DEFINITE INTEGRALS

In his earlier works, Cauchy used his theorem as a tool for calculating various definite integrals of functions of a real variable (mainly

improper integrals). To have an idea of these applications of the Cauchy theorem, we give here three examples.

1. *The Fresnel integrals :* $\int\limits_0^\infty \cos \xi^2\, d\xi$ and $\int\limits_0^\infty \sin \xi^2\, d\xi$.

For calculating these integrals which appear in the theory of diffraction, let us examine an auxiliary function of a complex variable, $F(z) = e^{iz^2}$. This function may be regarded as a composite function $F(z) = \varphi[f(z)]$, where $f(z) = iz^2$ and $\varphi(\zeta) = e^{\zeta}$. From this, according to the rules of differentiation of a composite function, it follows that $F(z)$ is differentiable in the whole plane and, moreover, $\frac{dF(z)}{dz} = 2ize^{iz^2}$. Consequently, the integral theorem of Cauchy is applicable to it.

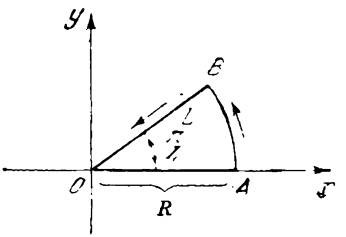


Fig. 36.

We take line L of Fig. 36 as the contour of integration. It consists of the segment OA of the positive part of the real axis, having a length R (R being any arbitrary positive number), the arc AB of a circle of radius R with its centre at the origin of coordinates and the segment BO of the bisector of the first quadrantal angle. The angle AOB is thus equal to $\frac{\pi}{4}$. By the integral theorem, the integral $\int\limits_L e^{i\zeta^2}\, d\zeta$ is equal to zero :

$$\int\limits_L e^{i\zeta^2}\, d\zeta = \int\limits_{OA} e^{i\zeta^2}\, d\zeta + \int\limits_{AB} e^{i\zeta^2}\, d\zeta + \int\limits_{BO} e^{i\zeta^2}\, d\zeta = 0.$$

On OA, ζ is equal to the real number ξ. Hence, $d\zeta = d\xi$ and

$$J_1(R) = \int\limits_{OA} e^{i\zeta^2}\, d\zeta = \int\limits_0^R e^{i\xi^2}\, d\xi.$$

On AB, $\zeta = R(\cos\varphi + i\sin\varphi)$, where φ varies from zero to $\pi/4$. Hence

$$\zeta^2 = R^2(\cos 2\varphi + i\sin 2\varphi) \qquad \left(0 \leqslant 2\varphi \leqslant \frac{\pi}{2}\right),$$

$$d\zeta = R(-\sin\varphi + i\cos\varphi)\, d\varphi = iR(\cos\varphi + i\sin\varphi)\, d\varphi$$

and

$$J_2(R) = \int\limits_{AB} e^{i\zeta^2}\, d\zeta = \int\limits_0^{\frac{\pi}{4}} \exp\left[iR^2(\cos 2\varphi + i\sin 2\varphi)\right] iR(\cos\varphi + i\sin\varphi)\, d\varphi.$$

Finally, on BO, $\zeta = \rho\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, where ρ varies from R to zero. Hence

$$\zeta^2 = \rho^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i\rho^2, \quad d\zeta = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)d\rho$$

and

$$J_3(R) = \int_{BO} e^{i\zeta^2}\,d\zeta = \int_R^0 e^{-\rho^2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)d\rho =$$

$$= -\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\int_0^R e^{-\rho^2}\,d\rho = -\frac{\sqrt{2}}{2}(1+i)\int_0^R e^{-\rho^2}\,d\rho.$$

Now, let R increase without limit. Then $J_3(R)$ will tend to the limit

$$-\frac{\sqrt{2}}{2}(1+i)\int_0^\infty e^{-\rho^2}\,d\rho = -\frac{\sqrt{2\pi}}{4}(1+i),$$

because

$$\int_0^\infty e^{-\rho^2}\,d\rho = \frac{\sqrt{\pi}}{2}.$$[1]

---

[1] In fact

$$\int_{-R}^{R}\int e^{-\xi^2-\eta^2}\,d\xi\,d\eta = \left(\int_{-R}^{R} e^{-\xi^2}\,d\xi\right)^2 = 4\left(\int_0^R e^{-\xi^2}\,d\xi\right)^2. \qquad (\alpha)$$

Let us inscribe, in a square with its centre at the origin of coordinates and its sides parallel to the coordinate axes and equal in length to $2R$, a circle k, and also circumscribe a circle K about this square. Then, by virtue of the integrand being positive,

$$\iint_k e^{-\xi^2-\eta^2}\,d\xi\,d\eta < \int_{-R}^{R}\int_{-R}^{R} e^{-\xi^2-\eta^2}\,d\xi\,d\eta < \iint_K e^{-\xi^2-\eta^2}\,d\xi\,d\eta$$

or, replacing the rectangular Cartesian coordinates ξ and η by polar coorditnates ρ and φ and using the formula (α), we have :

$$\int_0^{2\pi}\int_0^R e^{-\rho^2}\,\rho\,d\rho\,d\varphi < 4\left(\int_0^R e^{-\xi^2}\,d\xi\right)^2 < \int_0^{2\pi}\int_0^{R\sqrt{2}} e^{-\rho^2}\,\rho\,d\rho\,d\varphi.$$

Carrying out the integration and extracting the square root of all the members of the inequality, we get

$$\sqrt{\pi(1-e^{-R^2})} < 2\int_0^R e^{-\xi^2}\,d\xi < \sqrt{\pi(1-e^{-2R^2})},$$

whence

$$\lim_{R\to\infty} 2\int_0^R e^{-\xi^2}\,d\xi = \sqrt{\pi} \quad \text{or} \quad \int_0^\infty e^{-\xi^2}\,d\xi = \frac{\sqrt{\pi}}{2}.$$

We shall now show that $J_2(R) \to 0$ as $R \to \infty$. For this we estimate the modulus $|J_2(R)|$. One has

$$|J_2(R)| \leqslant R \int_0^{\frac{\pi}{4}} |\exp[iR^2(\cos 2\varphi + i \sin 2\varphi)]| \cdot |\cos\varphi + i \sin\varphi| \, d\varphi.$$

Here

$$|\exp[iR^2(\cos 2\varphi + i \sin 2\varphi)]| = \exp(-R^2 \sin 2\varphi) \text{ and } |\cos\varphi + i \sin\varphi| = 1;$$

and so

$$|J_2(R)| \leqslant R \int_0^{\frac{\pi}{4}} \exp(-R^2 \sin 2\varphi) \, d\varphi .$$

But $\sin 2\varphi \geqslant \frac{2}{\pi} \cdot 2\varphi$ for $0 \leqslant 2\varphi \leqslant \frac{\pi}{2}$.[1]

Hence

$$|J_2(R)| \leqslant R \int_0^{\frac{\pi}{4}} \exp\left(-\frac{4}{\pi} R^2\varphi\right) d\varphi = R \frac{\exp\left(-\frac{4}{\pi} R^2\varphi\right)}{-\frac{4}{\pi} R^2}\Bigg|_0^{\frac{\pi}{4}} =$$

$$= \frac{\pi}{4} \frac{1 - e^{-R^2}}{R},$$

whence it follows that

$$\lim_{R \to \infty} J_2(R) = 0.$$

Finally, we examine

$$J_1(R) = \int_0^R e^{i\xi^2} \, d\xi = \int_0^R \cos \xi^2 \, d\xi + i \int_0^R \sin \xi^2 \, d\xi.$$

---

[1] In fact, the function $f(\alpha) = \frac{\sin\alpha}{\alpha}$ decreases in the interval $\left(0, \frac{\pi}{2}\right)$ because its derivative

$$f'(\alpha) = \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = \frac{\cos\alpha(\alpha - \tan\alpha)}{\alpha^2} < 0$$

for $0 < \alpha < \frac{\pi}{2}$. Hence, $f(\alpha) > f\left(\frac{\pi}{2}\right)$ if $\alpha < \frac{\pi}{2}$, i.e. $\frac{\sin\alpha}{\alpha} > \frac{2}{\pi}$, or $\sin\alpha > \frac{2}{\pi}\alpha \quad \left(0 < \alpha < \frac{\pi}{2}\right)$.

This inequality becomes an equality for $\alpha = 0$ or $\alpha = \frac{\pi}{2}$.

Since $J_1(R) + J_2(R) + J_3(R) = 0$ for any R, we have: $J_1(R) = -J_2(R) - J_3(R)$ and

$$\lim_{R\to\infty} J_1(R) = -\lim_{R\to\infty} J_2(R) - \lim_{R\to\infty} J_3(R) = \frac{\sqrt{2\pi}}{4}(1+i),$$

i.e.

$$\lim_{R\to\infty}\left(\int_0^R \cos\xi^2\,d\xi + i\int_0^R \sin\xi^2\,d\xi\right) = \frac{\sqrt{2\pi}}{4}(1+i).$$

Hence it follows, firstly, that there do exist the integrals

$$\int_0^\infty \cos\xi^2\,d\xi = \lim_{R\to\infty}\int_0^R \cos\xi^2\,d\xi \text{ and } \int_0^\infty \sin\xi^2\,d\xi = \lim_{R\to\infty}\int_0^R \sin\xi^2\,d\xi$$

and, secondly, that

$$\int_0^\infty \cos\xi^2\,d\xi = \int_0^\infty \sin\xi^2\,d\xi = \frac{\sqrt{2\pi}}{4}.$$

2. *The integral*: $\int_{-\infty}^{+\infty} e^{-\lambda x^2}\cos(2\lambda\alpha x)\,dx \quad (\lambda>0, \quad \alpha>0).$

For calculating this integral let us integrate the function $f(z) = e^{-\lambda z^2}$ along the contour L of the rectangle drawn in Fig. 37. Since

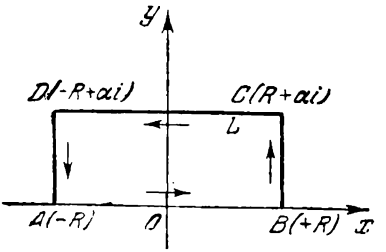


Fig. 37.

this function is differentiable in the whole plane, the integral theorem of Cauchy is applicable to it. Hence

$$\int_L e^{-\lambda\zeta^2}\,d\zeta = \int_{AB} e^{-\lambda\zeta^2}\,d\zeta + \int_{BC} e^{-\lambda\zeta^2}\,d\zeta + \int_{CD} e^{-\lambda\zeta^2}\,d\zeta + \int_{DA} e^{-\lambda\zeta^2}\,d\zeta = 0.$$

On AB

$$\zeta = x\,(-R \leqslant x \leqslant R) \quad \text{and} \quad d\zeta = dx\,;$$

hence,

$$J_1 = \int_{AB} e^{-\lambda\zeta^2}\, d\zeta = \int_{-R}^{+R} e^{-\lambda x^2} dx.$$

On BC

$$\zeta = R + iy \quad (0 \leqslant y \leqslant \alpha), \qquad \zeta^2 = R^2 + 2iRy - y^2 \text{ and } d\zeta = i\,dy\,;$$

hence,

$$J_2 = \int_{BC} e^{-\lambda\zeta^2}\, d\zeta = \int_0^{\alpha} \exp\left[-\lambda\,(R^2+i2Ry-y^2)\right] i\,dy =$$

$$= i \int_0^{\alpha} \exp\left[-\lambda\,(R^2-y^2)\right] \exp\left(-2Riy\lambda\right) dy,$$

On CD

$$\zeta = x + i\alpha \quad (R \geqslant x \geqslant -R), \qquad \zeta^2 = x^2 + 2i\alpha x - \alpha^2 \text{ and } d\zeta = dx\,;$$

hence,

$$J_3 = \int_{CD} e^{-\lambda\zeta^2}\, d\zeta = \int_R^{-R} \exp\left[-\lambda\,(x^2+2i\alpha x-\alpha^2)\right] dx =$$

$$= -\,e^{\lambda\alpha^2} \int_{-R}^{+R} \exp\left(-\lambda x^2 - 2i\alpha\lambda x\right) dx =$$

$$= -\,e^{-\lambda\alpha^2} \int_{R-}^{+R} e^{-\lambda x^2} \left[\cos\left(2\lambda\alpha x\right) - i \sin\left(2\lambda\alpha x\right)\right] dx.$$

Finally, on DA

$$\zeta = -R + iy \quad (\alpha \geqslant y \geqslant 0), \qquad \zeta^2 = R^2 - 2Riy - y^2 \text{ and } d\zeta = i\,dy\,;$$

hence

$$J_4 = \int_{DA} e^{-\lambda\zeta^2}\, d\zeta = \int_{\alpha}^{0} \exp\left[-\lambda\,(R^2-2Riy-y^2)\right] i\,dy =$$

$$= -\,i \int_0^{\alpha} \exp\left[-\lambda\,(R^2-y^2)\right] \exp\left(2Riy\lambda\right) dy.$$

Now, let R increase without limit. Then the integrals J_2 and J_4 tend to zero. In fact

$$|J_2| \leqslant \int_0^{\alpha} |\exp[-\lambda(R^2-y^2)]| \cdot |\exp(-2iR\lambda y)|\, dy = \int_0^{\alpha} e^{-\lambda(R^2-y^2)}\, dy.$$

For $R > \alpha$, we obtain

$$|J_2| \leqslant \int_0^{\alpha} \exp[-\lambda(R^2-\alpha^2)]\, dy = \alpha \exp[-\lambda(R^2-\alpha^2)] \to 0 \quad (R \to \infty).$$

Exactly in the same manner, we get

$$|J_4| \leqslant \int_0^{\alpha} |\exp[-\lambda(R^2-y^2)]|\,|\exp 2iR\lambda y|\, dy =$$

$$= \int_0^{\alpha} \exp[-\lambda(R^2-y^2)]\, dy \to 0 \qquad (R \to \infty).$$

For $R \to \infty$, the integral J_1 tends to the limit

$$\int_{-\infty}^{+\infty} \exp(-\lambda x^2)\, dx = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \exp[-(\sqrt{\lambda}x)^2]\, d(\sqrt{\lambda}x) = \sqrt{\frac{\pi}{\lambda}}$$

$\left(\text{since } \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}\text{; see the previous example}\right)$.

Finally, from the relation

$$J_1 + J_2 + J_3 + J_4 = 0$$

we derive

$$e^{\lambda\alpha^2} \int_{-\infty}^{+\infty} e^{-\lambda x^2} [\cos(2\lambda\alpha x) - i \sin(2\lambda\alpha x)]\, dx =$$

$$= -\lim_{R\to\infty} J_3 = \lim_{R\to\infty} J_1 + \lim_{R\to\infty} J_2 + \lim_{R\to\infty} J_4 = \sqrt{\frac{\pi}{\lambda}}.$$

Comparing the real parts, we obtain

$$e^{\lambda\alpha^2}\int_{-\infty}^{+\infty} e^{-\lambda x^2}\cos(2\lambda\alpha x)\,dx = \sqrt{\frac{\pi}{\lambda}},$$

i.e.

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2}\cos(2\lambda\alpha x)\,dx = \sqrt{\frac{\pi}{\lambda}}\,e^{-\lambda\alpha^2}.$$

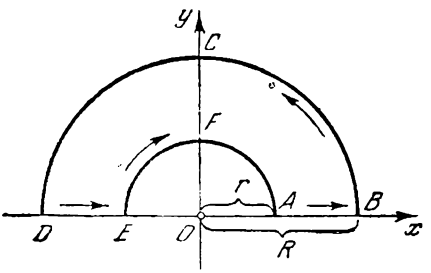


Fig. 38.

3. *Integral* $\int_0^{\infty}\frac{\sin\xi}{\xi}\,d\xi$.

We take the auxiliary function $f(z)=\frac{e^{iz}}{z}$. This function is defined in the domain G, formed by all the points of the plane except the origin of coordinates, and is differentiable in this domain.

Taking the contour of integration L as shown in Fig. 38 (the contour as well as all points of its interior lie in the simply-connected domain D, bounded by the negative part of the imaginary axis), we derive by applying the integral theorem to $f(z)=\frac{e^{iz}}{z}$:

$$\int_L \frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_{AB}\frac{e^{i\zeta}\,d\zeta}{\zeta} + \int_{BCD}\frac{e^{i\zeta}\,d\zeta}{\zeta} + \int_{DE}\frac{e^{i\zeta}\,d\zeta}{\zeta} + \int_{EFA}\frac{e^{i\zeta}\,d\zeta}{\zeta} = 0.$$

On AB, ζ is equal to the real number ξ. Hence $d\zeta = d\xi$ and the integral

$$J_1 = \int_{AB}\frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_r^R \frac{e^{i\xi}\,d\xi}{\xi} = \int_r^R \frac{\cos\xi\,d\xi}{\xi} + i\int_r^R \frac{\sin\xi\,d\xi}{\xi}.$$

On BCD, $\zeta = R(\cos\varphi + i\sin\varphi)$ $(0\leqslant\varphi\leqslant\pi)$; hence

$$d\zeta = R(-\sin\varphi + i\cos\varphi)\,d\varphi = iR(\cos\varphi + i\sin\varphi)\,d\varphi$$

and

$$J_2 = \int_{BCD}\frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_0^{\pi}\frac{\exp[iR(\cos\varphi + i\sin\varphi)]\,iR(\cos\varphi + i\sin\varphi)\,d\varphi}{R(\cos\varphi + i\sin\varphi)} =$$

$$= i\int_0^{\pi}\exp(iR\cos\varphi - R\sin\varphi)\,d\varphi.$$

On DE, ζ is equal to the real number ξ; hence $d\zeta = d\xi$ and

$$J_3 = \int_{DE} \frac{e^{i\zeta}\, d\zeta}{\zeta} = \int_{-R}^{-r} \frac{e^{i\xi}\, d\xi}{\xi} = \int_{-R}^{-r} \frac{\cos \xi\, d\xi}{\xi} + i \int_{-R}^{-r} \frac{\sin \xi\, d\xi}{\xi}.$$

Replacing in this the variable of integration ξ by $-\xi$, we obtain

$$J_3 = -\int_{r}^{R} \frac{\cos \xi\, d\xi}{\xi} + i \int_{r}^{R} \frac{\sin \xi\, d\xi}{\xi}.$$

Finally, on EFA

$$\zeta = r(\cos \varphi + i \sin \varphi) \quad (\pi \geqslant \varphi \geqslant 0);$$

hence

$$d\zeta = ir(\cos \varphi + i \sin \varphi)\, d\varphi$$

and

$$J_4 = \int_{EFA} \frac{e^{i\zeta}\, d\zeta}{\zeta} = \int_{\pi}^{0} \frac{\exp [ir(\cos \varphi + i \sin \varphi)]\, ir(\cos \varphi + i \sin \varphi)}{r(\cos \varphi + i \sin \varphi)}\, d\varphi =$$

$$= -i \int_{0}^{\pi} \exp (ir \cos \varphi - r \sin \varphi)\, d\varphi.$$

Now, let R increase without limit, whereupon

$$J_2 = i \int_{0}^{\pi} \exp (iR \cos \varphi - R \sin \varphi)\, d\varphi$$

will tend to zero. In fact,

$$|J_2| = \left| \int_{0}^{\pi} \exp (iR \cos \varphi - R \sin \varphi)\, d\varphi \right| \leqslant \int_{0}^{\pi} \exp (-R \sin \varphi)\, d\varphi =$$

$$= \int_{0}^{\pi/2} \exp (-R \sin \varphi)\, d\varphi < 2 \int_{0}^{\pi/2} \exp\left(-R \frac{2}{\pi} \varphi\right) d\varphi =$$

$$= \pi \frac{1 - e^{-R}}{R} < \frac{\pi}{R},$$

whence it follows that $\lim_{R \to \infty} J_2 = 0$.

Finally, let r tend to zero and find the limit of the integral

$$J_4 = -i \int_0^{\pi} \exp(ir\cos\varphi - r\sin\varphi)\, d\varphi.$$

Since the function $\exp(iz)$ is continuous at the point $z = 0$ where it reduces to unity, hence we can, for any $\varepsilon > 0$, find such $\delta(\varepsilon) > 0$ that for $|z| = r < \delta(\varepsilon)$ the inequality

$$|\exp(iz) - 1| = |\exp(ir\cos\varphi - r\sin\varphi) - 1| < \varepsilon$$

is satisfied. Hence it follows that

$$\left| J_4 - \left(-i \int_0^{\pi} 1 \cdot d\varphi\right) \right| = \left| \int_0^{\pi} [\exp(i\cos\varphi - r\sin\varphi) - 1]\, d\varphi \right| < \pi\varepsilon,$$

that is,

$$\lim_{r\to 0} J_4 = -i \int_0^{\pi} 1 \cdot d\varphi = -\pi i.$$

Turning to the fundamental relation

$$\int_L \frac{e^{i\zeta}\, d\zeta}{\zeta} = J_1 + J_2 + J_3 + J_4 = 0,$$

we obtain $J_1 + J_3 = -J_2 - J_4$ or, using the expressions mentioned above for J_1 and J_3,

$$2i \int_r^R \frac{\sin\xi\, d\xi}{\xi} = -J_2 - J_4.$$

As $R \to \infty$ and $r \to 0$, the right-hand side, as we have already seen, tends to the limit $-\lim\limits_{R\to\infty} J_2 - \lim\limits_{r\to 0} J_4 = \pi i$. Hence, the left-hand side also tends to the same limit:

$$\lim_{\substack{R\to\infty \\ r\to 0}} 2i \int_r^R \frac{\sin\xi}{\xi}\, d\xi = \pi i.$$

Denoting $\lim\limits_{\substack{R\to\infty \\ r\to 0}} \int\limits_r^R \frac{\sin\xi}{\xi}\,d\xi$ (the existence of which has already been proved) by $\int\limits_0^\infty \frac{\sin\xi}{\xi}\,d\xi$, we finally obtain

$$\int_0^\infty \frac{\sin\xi}{\xi}\,d\xi = \frac{\pi}{2}.$$

7. INTEGRAL AND THE PRIMITIVE

From the integral theorem of Cauchy it follows that the integrals of an analytic function along any two curves L_1 and L_2, with a common origin z_0 and end z, (Fig. 39), have equal values. In fact, the curve $L = L_1 + (-L_2)$ is a closed curve and, hence,

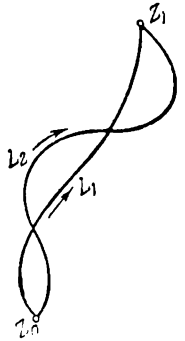


Fig. 39.

$$\int_L f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{-L_2} f(z)\,dz =$$
$$= \int_{L_1} f(z)\,dz - \int_{L_2} f(z)\,dz = 0,$$

whence it follows that

$$\int_{L_1} f(z)\,dz = \int_{L_2} f(z)\,dz.$$

Thus, *the integral of a function $f(z)$, analytic in a simply-connected domain G, depends only on the initial and the final points of the path of integration.* Hence, for an integral along any arbitrary rectifiable curve L, connecting the points z_0 and z, we can use the symbol $\int\limits_{z_0}^{z} f(z)\,dz$. We shall call z_0 and z the *limits of integration* (the lower and the upper limit, respectively).

Fixing z_0, we examine the integral $\int\limits_{z_0}^{z} f(z)\,dz$ as a function of the upper limit :

$$\int_{z_0}^{z} f(z)\,dz = F(z).$$

We will show that $F(z)$ is an *analytic function and that its derivative is equal to* $f(z)$. For future convenience we give a more general formulation of this proposition.

Theorem. *Let $f(z)$ be a function continuous in the simply-connected domain G, for which the integral along any rectifiable curve, lying in the domain, depends only on the initial and final points of the curve. Then the integral*

$$F(z) = \int_{z_0}^{z} f(z)\, dz$$

is an analytic function; moreover,

$$F'(z) = f(z).$$

Proof. We examine any neighbourhood U of the point z: $|\zeta - z| < \rho$, lying in the domain G. Let $L \subset G$ be any curve connecting z_0 with z, and let λ be a rectilinear segment connecting z with any point $z_1 \neq z$, $z_1 \subset U$ (Fig. 40). Then

$$F(z) = \int_{L} f(z)\, dz, \qquad F(z_1) = \int_{L+\lambda} f(z)\, dz = \int_{L} f(z)\, dz + \int_{\lambda} f(z)\, dz$$

and, consequently,

$$F(z_1) - F(z) = \int_{\lambda} f(\zeta)\, d\zeta,$$

$$\frac{F(z_1)-F(z)}{z_1-z} - f(z) = \frac{\int_{\lambda} f(\zeta)\, d\zeta - f(z)\,(z_1-z)}{z_1-z} = \frac{\int_{\lambda} [f(\zeta)-f(z)]\, d\zeta}{z_1-z}.$$

Due to the continuity of $f(z)$ we can, for any $\varepsilon > 0$, choose the radius ρ of the neighbourhood U so small that for any point $\zeta \in U$ the inequality $|f(\zeta) - f(z)| < \varepsilon$ is satisfied. Then we will have

$$\left| \frac{F(z_1)-F(z)}{z_1-z} - f(z) \right| < \frac{\varepsilon \cdot \text{length } \lambda}{|z_1-z|} = \varepsilon$$

(because $|z_1 - z| = \text{length } \lambda$); hence,

$$\lim_{z_1 \to z} \frac{F(z_1)-F(z)}{z_1-z} = f(z) = F'(z).$$

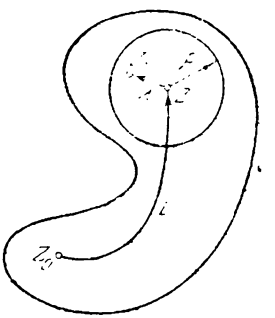


Fig. 40.

We call the function $\Phi(z)$, analytic in some domain G, the *primitive with respect to the function* $f(z)$ [in short, the primitive of $f(z)$] if $\Phi'(z) = f(z)$ at all points of the domain. Under the conditions of the theorem proved above, we can state that $\int_{z_0}^{z} f(z)\,dz$ is the primitive of $f(z)$. We now show that any primitive of $f(z)$ is expressible by the formula

$$\Phi(z) = \int_{z_0}^{z} f(z)\,dz + C, \tag{1}$$

where C is a constant. We now put

$$\omega(z) = \Phi(z) - \int_{z_0}^{z} f(z)\,dz = u(x, y) + iv(x, y);$$

then

$$\omega'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f(z) - f(z) \equiv 0,$$

whence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} \equiv 0$$

and, consequently,

$$u(x, y) = C_1, \quad v(x, y) = C_2,$$

that is,

$$\omega(z) = u(x, y) + iv(x, y) \equiv C_1 + iC_2 = C.$$

Putting $z = z_0$ in the formula (1), we get

$$\Phi(z_0) = C;$$

hence

$$\int_{z_0}^{z} f(z)\,dz = \Phi(z) - \Phi(z_0). \tag{2}$$

This result enables us to reduce the evaluation of the integral of an analytic function $f(z)$ to the calculation of any primitive of the function.

Thus, for instance, observing that the function $\ln z = \ln |z| + i \arg z$ is the primitive of $f(z) = 1/z$ in the domain $-\pi < \arg z < \pi$, we obtain

$$\int_1^z \frac{dz}{z} = \ln z - \ln 1 = \ln z.$$

8. GENERALISATION OF THE INTEGRAL THEOREM OF CAUCHY WHEN THE FUNCTION IS NOT ANALYTIC ON THE CONTOUR OF INTEGRATION

The integral theorem of Cauchy is generalised in different directions. *Let the domain G represent the interior of a rectifiable Jordan curve Γ and let $f(z)$ be a function continuous in the closed domain $\overline{G}$ and analytic in the domain G.* Then we can assert that $\int_\Gamma f(z)\,dz = 0$. Here, the generalisation means that $f(z)$ is not already assumed to be analytic at the points of the curve of integration. We shall prove this generalisation under the additional condition that every ray emanating from some point $z_0 \in G$, meets L only at one point and that L is composed of a finite number of smooth arcs. These conditions are evidently fulfilled when L is a convex polygon (particularly, a triangle) or a circle.

Let $z = z_0 + \lambda(t)$ $(0 \leqslant t \leqslant 2\pi)$ be the equation of L; $\lambda(t)$, by assumption, possesses a piecewise continuous derivative $\lambda'(t)$. We subject L to a similarity transformation in respect of ρ $(0<\rho<1)$ with centre at z_0. We thus obtain the closed curve $L_\rho : z = z_0 + \rho\lambda(t)$ $(0 \leqslant t \leqslant 2\pi)$, lying in the domain G. By the integral theorem of Cauchy

$$\int_{L_\rho} f(z)\,dz = \int_0^{2\pi} f[z_0 + \rho\lambda(t)]\,\rho\lambda'(t)\,dt = 0,$$

whence

$$\int_0^{2\pi} f[z_0 + \rho\lambda(t)]\,\lambda'(t)\,dt = 0.$$

Consequently,

$$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_0^{2\pi} f[z_0+\lambda(t)]\,\lambda'(t)\,dt\right| =$$

$$= \left|\int_0^{2\pi} \{f[z_0+\lambda(t)] - f[z_0+\rho\lambda(t)]\}\,\lambda'(t)\,dt\right| \leqslant$$

$$\leqslant \int_0^{2\pi} |f[z_0+\lambda(t)] - f[z_0+\rho\lambda(t)]|\,|\lambda'(t)|\,dt.$$

But, $f(z)$ is uniformly continuous in the closed domain $\bar{G}$; hence, for any $\varepsilon > 0$,

$$|f(z') - f(z'')| < \varepsilon, \text{ if } |z' - z''| < \delta(\varepsilon), \quad z', z'' \in \bar{G}.$$

We denote the upper limits of the functions $\lambda(t)$ and $\lambda'(t)$ on the segment $[0, 2\pi]$ by λ and λ', respectively. Then, $|[z_0+\lambda(t)] - [z_0+\rho\lambda(t)]| \leqslant (1-\rho)\lambda < \delta(\varepsilon)$ if $1 - \rho < \delta(\varepsilon)/\lambda$, and consequently

$$\left| \int_L f(z)\, dz \right| < \varepsilon \cdot \lambda' \cdot 2\pi.$$

Due to the arbitrariness of ε, it thus follows that

$$\int_L f(z)\, dz = 0.$$

9. THE THEOREM ON A COMPOSITE CONTOUR

Now, let $f(z)$ be a function, analytic in the domain G which is not a simply-connected domain in the finite plane. If L be a closed rectifiable curve lying in the domain G, then its interior may either lie within or outside the domain G. In the first case, all the arguments advanced above in the proof of the integral theorem of Cauchy are applicable without any alteration to the integral $\int_L f(z)\, dz$ and we once again obtain $\int_L f(z)\, dz = 0$. In the second case, these arguments cannot be applied to their logical end.

In fact, now we cannot even assert that the interior of any polygon inscribed in L and also the interior of any of the triangles into which this polygon can be decomposed belong to the domain G. Consequently, we cannot depend on the existence of a derivative of the function $f(z)$ at any point ζ of such a triangle (*cf.* Art. 5 of this chapter).

The simple example of the function $f(z) = 1/z$, analytic in the domain G, obtained from the finite plane by removing the origin of coordinates (this domain is not simply-connected in the finite plane), shows that in this case there exist closed rectifiable curves L along which $\int_L f(z)\, dz \neq 0$. Every circle with its centre at the origin of coordinates will be such a curve, because $\int_{|z|=\rho} \frac{dz}{z} = 2\pi i \neq 0$ [*cf.* Example (*c*) of Art. 1].

We shall now show with the help of the integral theorem of Cauchy that in any general case of the function $f(z)$, which is analytic in any

domain G (multiply-connected, in general), we can reduce the calculation of the integral along any closed contour to that of integrals along other closed contours lying in the interior of the former. To be more precise, *let Γ be a closed rectifiable Jordan curve lying in the interior of the domain G and let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be curves having the same properties and lying in the interior of Γ. We further require that every curve γ_j lies in the exterior of any other curve γ_k $(j, k = 1, 2, \ldots, n, j \neq k)$ and, finally, that the multiply-connected domain D, bounded by the curves $\Gamma, \gamma_1, \ldots$ and γ_n, is contained in G* (Fig. 41). *Under these conditions*

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz, \tag{3}$$

where all the paths of integration are directed in the same sense, for example, anti-clockwise.

To prove this we connect by rectifiable Jordan arcs, for instance, by broken lines, the curve Γ with γ_1, γ_1 with $\gamma_2, \ldots, \gamma_{n-1}$ with γ_n and γ_n with Γ such that no pair of these arcs $\delta_1, \delta_2, \ldots, \delta_{n+1}$ has points in common and that these are contained in the domain D (excluding the ends of the arcs, lying on the boundary of D). The initial and final points of the arcs $\delta_1, \delta_2, \ldots, \delta_{n+1}$ divide each of the curves $\Gamma, \gamma_1, \ldots, \gamma_n$ into two arcs ; we denote these by the same symbol as used for the entire curve but with one or two primes above. These primes are so put that the interior of each of the two closed Jordan curves composed respectively of the arcs Γ', δ_1, γ_1', δ_2, γ_2', $\ldots$, γ_n', δ_{n+1}, and Γ'', δ_{n+1}, γ_n'', δ_n, $\ldots$, γ_1'', δ_1, lies in the domain D and hence also in the domain G. Integrating $f(z)$ along

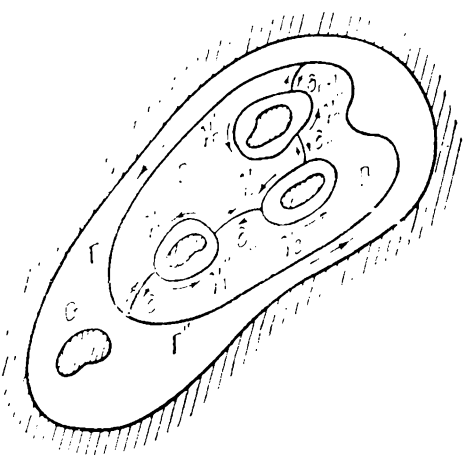


Fig. 41.

these closed curves in directions corresponding to the direction chosen for traversing the curve Γ (see Fig. 41) and noting that in this case γ_k, γ''

will be traversed in a direction opposite to that chosen for γ_k ($k=1, 2, \ldots, n$), we obtain

$$\int_{\Gamma'+\delta_1-\gamma_1'+\delta_2-\gamma_2'+\ldots-\gamma_n'+\delta_{n+1}} f(z)\,dz = \int_{\Gamma'} f(z)\,dz + \int_{\delta_1} f(z)\,dz -$$

$$-\int_{\gamma_1'} f(z)\,dz + \int_{\delta_2} f(z)\,dz - \int_{\gamma_2'} f(z)\,dz + \ldots - \int_{\gamma_n'} f(z)\,dz + \int_{\delta_{n+1}} f(z)\,dz = 0,$$

$$\int_{\Gamma''-\delta_{n+1}-\gamma_n''-\ldots-\delta_2-\gamma_1''-\delta_1} f(z)\,dz =$$

$$= \int_{\Gamma''} f(z)\,dz - \int_{\delta_{n+1}} f(z)\,dz - \int_{\gamma_n''} f(z)\,dz - \ldots -$$

$$- \int_{\delta_2} f(z)\,dz - \int_{\gamma_1''} f(z)\,dz - \int_{\delta_L} f(z)\,dz = 0.$$

We sum up these equalities termwise. Then, the integrals along the arcs $\delta_1, \ldots, \delta_{n+1}$ mutually cancel and the integrals along the arcs Γ' and Γ'', γ_1' and γ_1'', $\ldots$, γ_n' and γ_n'', taken pairwise, give integrals along the coresponding closed curves. Hence

$$\int_{\Gamma} f(z)\,dz - \int_{\gamma_1} f(z)\,dz - \ldots - \int_{\gamma_n} f(z)\,dz = 0, \qquad (3')$$

as was to be proved.

We now turn to the function $f(z) = \frac{1}{z}$ in the domain $G: z \neq 0$. If Γ be any closed rectifiable Jordan curve containing the origin of coordinates in its interior and γ be a circle with its centre at the origin and contained in the interior of Γ, then, as proved above,

$$\int_{\Gamma} \frac{1}{z}\,dz = \int_{\gamma} \frac{1}{z}\,dz = 2\pi i.$$

The result arrived at may be given a different form. We consider the boundary of the multiply-connected domain D, composed of the isolated curves $\Gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ as making one composite contour Γ. The direction for which the external contour Γ is traversed in the anticlockwise sense and the internal contours $\gamma_1, \ldots, \gamma_n$ in the clockwise

sense is regarded as the positive direction of the curve L. In other words, when the composite contour L is traversed in the positive direction, the interior D of this contour must remain to the left of the person traversing it. Now, the expression on the left-hand side of the equality (3') may be conventionally written as $\int_L f(z)\,dz$. Then, we can state *that the integral of $f(z)$ along the composite contour L, lying in G (along with the domain D bounded by this contour), is equal to zero.*

This proposition is evidently an extension of the integral theorem of Cauchy to the case of a composite contour belonging to multiply-connected domains. For convenience of reference, we shall call the proposition proved here the *theorem on a composite contour.*

10. INTEGRAL AS A FUNCTION OF A POINT IN A MULTIPLY-CONNECTED DOMAIN

The theorem on a composite contour enables us to study the integrals of analytic functions in a multiply-connected domain. Let G be a multiply-connected domain and $f(z)$ a function which is one-valued and analytic in it. If $\int_L f(z)\,dz = 0$ for any closed rectifiable curve belonging to G (this will be the case, for instance, for the function $f(z) = \frac{1}{1+z^2}$ in the circular ring $1 < |z| < R$, where R is any number greater than unity), then we can apply the arguments of Art. 7 to the integral $\int_{z_1}^{z} f(z)\,dz$. We again find that this integral is a one-valued analytic function—the primitive of $f(z)$. We shall now assume that there exist in the domain G closed contours L (at least one) for which $\int_L f(z)\,dz \neq 0$ (this will be true, for example, for the function $\frac{1}{z}$ in the domain $z \neq 0$ or for the function $\frac{1}{1+z^2}$ in the domain $1 + z^2 \neq 0$, i.e. $z \neq i$ and $z \neq -i$; in the first case $\int_{|z|=\rho} \frac{1}{z}\,dz = 2\pi i$; in the second case, for $r < 2$, $\int_{|z-i|=r} \frac{dz}{1+z^2} = \pi$ and $\int_{|z+i|=r} \frac{dz}{1+z^2} = -\pi$). Then, for the two

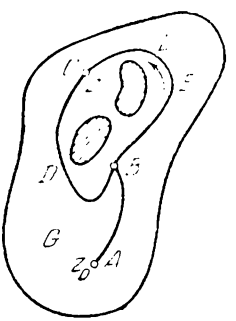


Fig. 42.

curves $ABEC$ and $ABDC$ (Fig. 42), connecting the points $A(z_0)$ and $C(z)$, we have

$$\int_{ABEC} f(z)\,dz - \int_{ABDC} f(z)\,dz = \int_{BEC} f(z)\,dz - \int_{BDC} f(z)\,dz = \int_{L} f(z)\,dz \neq 0;$$

consequently, the integral $\int_{z_0}^{z} f(z)\,dz$ has at least two different values at the point z, i.e. the function $F(z) = \int_{z_0}^{z} f(z)\,dz$ is many-valued in the domain G.

We shall explain the character of such a many-valuedness by the example of the function $f(z) = 1/z$ in the domain $G: z \neq 0$. We shall find that the integral $\int_{L} \frac{1}{z}\,dz$, where $L \subset G$ is an arbitrary, rectifiable curve connecting the points $z_0 = 1$ and z, may be represented in the form of a sum of a integral along any other arbitrary curve l (connecting the same points) and an integral multiple of the value of the integral along a closed Jordan contour γ containing the point 0 in its interior (this point not belonging to the domain). Let L, l and γ be contours shown in Fig. 43. We join to L the arcs AB and AC which are traversed twice (in opposite directions) during integration.

We obtain

$$\int_{L} \frac{dz}{z} = \int_{AaBA} \frac{dz}{z} + \int_{ABbCA} \frac{dz}{z} = \int_{ACD} \frac{dz}{z}.$$

Each of the two closed contours, along which the first two integrals of the right-hand side are taken, forms together with the contour γ a composite contour. Hence, by the theorem of Art. 9,

$$\int_{AaBA} \frac{dz}{z} = \int_{\gamma} \frac{dz}{z} \quad \text{and} \quad \int_{ABbCA} \frac{dz}{z} = \int_{\gamma} \frac{dz}{z}.$$

Further, the curves l and DCA form a closed contour the interior of which lies in the domain G; thus, the integral along it is equal to zero.

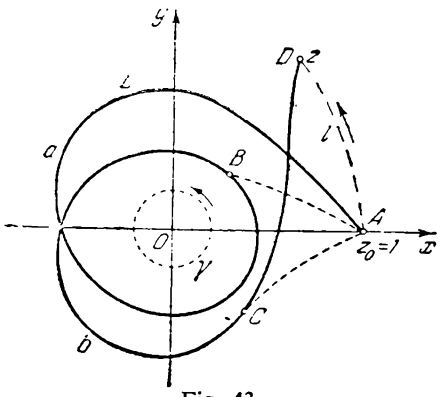


Fig. 43.

Hence

$$\int_{ACD} \frac{dz}{z} = \int_{l} \frac{dz}{z}.$$

Thus

$$\int_{L} \frac{dz}{z} = \int_{l} \frac{dz}{z} + 2\int_{\gamma} \frac{dz}{z}.$$

This reasoning is of a completely general character. For any arbitrary curve L we have

$$\int_{L} \frac{dz}{z} = \int_{l} \frac{dz}{z} + n\int_{\gamma} \frac{dz}{z},$$

where n is some integer which may be equal to zero and may also be negative (this will be so when the closed contours constructed according to the previous reasoning are directed negatively). We can, therefore, always find a curve L such that n has any preassigned (integral) value.

We can choose the circle $|z| = \rho$ as the contour γ. Hence, $\int_{\gamma} \frac{dz}{z} = 2\pi i$. Further, if the point z does not lie on the negative part of the real axis ($|\arg z| < \pi$), we can take as l the curve lying in the domain $|\arg z| < \pi$. Then we derive $\int_{l} \frac{dz}{z} = \ln z$ (*cf.* Art. 7). If, on the other hand, $\arg z = \pi$, then for l we can take a curve lying in the domain $\arg z \neq -\pi/2$ (i.e. not intersecting the negative part of the imaginary axis). We shall denote by θ the value of the argument of any point of this domain, a value lying between $-\pi/2$ and $3\pi/2$. Then $\theta = 0$ for the initial point of l and $\theta = \pi$ for the final point. Since the function $\varphi(z) = \ln|z| + i\theta$ in the simply-connected domain $\arg z \neq -\pi/2$ is a primitive of $1/z$, therefore, by Art. 7

$$\int_{l} \frac{dz}{z} = \varphi(z) - \varphi(1) = \ln|z| + i\pi = \ln z.$$

Thus, in all cases

$$F(z) = \int_{1}^{z} \frac{dz}{z} = \int_{L} \frac{dz}{z} = \ln z + 2n\pi i = \operatorname{Ln} z.$$

We find that the integral $\int_1^z \frac{dz}{z}$ represents a many-valued function Ln z in the domain $z \neq 0$. We leave it to the reader to prove that the integral $\int_0^z \frac{dz}{1+z^2}$, in the domain $z \neq \pm i$, also represents the many-valued function $\operatorname{Arctan} z = \frac{1}{2i} \operatorname{Ln} \frac{i-z}{i+z}$.

6

THE CAUCHY INTEGRAL FORMULA AND ITS CONSEQUENCES

1. THE CAUCHY INTEGRAL FORMULA

Let $f(z)$ be a function which is one-valued and analytic in the domain G and L be a closed rectifiable Jordan curve belonging to G, along with its interior D, (Fig. 44). Then, for any point $z_0 \in D$, the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0} \tag{1}$$

holds.

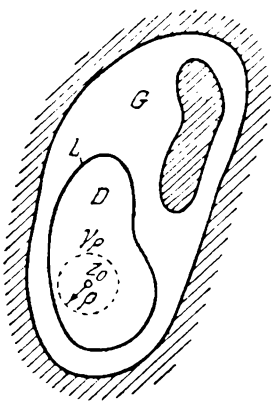


Fig. 44.

In this formula the curve L is traversed in the positive direction, i. e. anti-clockwise. This formula expresses the values of an analytic function in the interior of a closed curve in terms of the values of the same function on the curve itself.

For proving this, we draw a circle γ_ρ with its centre at the point z_0 and of radius ρ so small that the circle $|z - z_0| \leqslant \rho$ lies in the interior of L. Then, for the composite contour formed by the curves L and γ_ρ, we will have :

$$\frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)\,dz}{z-z_0}. \tag{2}$$

Consequently, for proving the formula (1), it is sufficient to establish the equality

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)\,dz}{z-z_0}$$

or

$$\int_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) =$$

$$= \int_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_{\gamma_\rho} \frac{dz}{z-z_0} = \int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0. \tag{3}$$

By virtue of the continuity of $f(z)$ at the point z_0, the inequality

$$|f(z)-f(z_0)| < \varepsilon \quad (z \in \gamma_\rho)$$

will be satisfied for any $\varepsilon > 0$, if $\rho < \delta(\varepsilon)$. With this condition, we get

$$\left| \int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < \frac{\varepsilon}{\rho}\, 2\pi\rho = 2\pi\varepsilon.$$

Consequently

$$\lim_{\rho \to 0} \int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0. \tag{4}$$

But the integral here is equal to the left-hand side of the equality (3) and hence does not depend on ρ [that the integral $\int_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz$ does not depend on ρ is clear from the equality (2)]. Consequently, it is equal to zero for all values of ρ in question. Thus, the equality (3) holds and along with it does the Cauchy integral formula (1).

The integral $J(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)}{z-z_0}\,dz$ is called the *Cauchy integral.*

We have calculated its value for z_0 lying in the interior of the curve L. If ^0z lies in the exterior E of the curve L, then the integrand $\frac{f(z)}{z-z_0}$ is analytic not only on L but also everywhere in the interior of L (the divisor $z - z_0$ is different from zero on L and in the interior of L). According to the Cauchy integral theorem, this integral is equal to zero in this case. Thus

$$J(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0} = \begin{cases} f(z_0) & (z_0 \in D), \\ 0 & (z_0 \in E). \end{cases} \tag{5}$$

For $z_0 \in L$, the Cauchy integral, generally speaking, loses its meaning not only as a proper but also as an improper integral.

As a special case, let L be a circle with its centre at z_0. If ρ is the radius of the circle then its equation may be written in the form

$$z = z_0 + \rho e^{i\varphi} \quad (0 \leqslant \varphi \leqslant 2\pi);$$

Hence

$$f(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\varphi})\, i\rho e^{i\varphi}}{\rho e^{i\varphi}}\, d\varphi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\varphi})\, d\varphi. \qquad (6)$$

The last integral may be looked upon as the arithmetic mean of the values of $f(z)$ on the circle $|z - z_0| = \rho$. In fact, this integral is equal to the limit, as $n \to \infty$, of the arithmetic mean of the values of $f(z)$ at the vertices of any regular n-gon inscribed in the circle:

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\varphi})\, d\varphi = \lim_{n\to\infty} \frac{1}{2\pi} \sum_{k=0}^{n-1} f\left(z_0+\rho e^{i\left(\varphi_0 + \frac{2k\pi}{n}\right)}\right)\frac{2\pi}{n} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(z_0+\rho e^{i\left(\varphi_0 + \frac{2k\pi}{n}\right)}\right).$$

Thus, *the value of an analytic function $f(z)$ at any point z_0 of its domain of analyticity is equal to the arithmetic mean of its values on any circle with centre z_0 (it is only necessary that the circle itself and its interior lie in the same domain).*

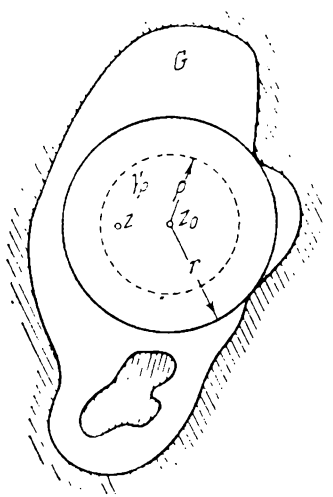


Fig. 45.

2. DECOMPOSITION OF AN ANALYTIC FUNCTION INTO A POWER SERIES. THE THEOREM OF LIOUVILLE

It has been shown in Art. 3 (Chapter 4) that the sum of a power series is an analytic function within the circle of convergence of the series. Now, we can show with the help of the Cauchy integral formula that any analytic function in a circle may be represented in the form of a power series. To be precise, we shall prove the following theorem.

Theorem. *Let $f(z)$ be a function which is one-valued and analytic in the domain G. If $z_0 \in G$ and r is the distance of z_0 from the boundary of G, then in the circle $|z - z_0| < r$, $f(z)$ can be decomposed in terms of the powers of $z - z_0$.*

Let z be a point in the circular domain $|z - z_0| < r$. We now examine a concentric circle of radius ρ $(0<\rho<r)$, containing this point in its interior (Fig. 45).

If γ_ρ is the boundary of this circle, then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{7}$$

To prove the theorem it is sufficient to decompose $\frac{f(\zeta)}{\zeta - z}$ into a series in the powers of $(z - z_0)$ and then integrate it term by term. We have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_0^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

and

$$\frac{f(\zeta)}{\zeta - z} = \sum_0^\infty f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

For a fixed value of z, the last series converges uniformly with respect to $\zeta \in \gamma_\rho$, because $\left| f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \right| \leqslant \max_{\gamma_\rho} |f(\zeta)| \cdot \frac{|z - z_0|^n}{\rho^{n+1}}$, and the number series $\sum_0^\infty \max_{\gamma_\rho} |f(\zeta)| \cdot \frac{|z - z_0|^n}{\rho^{n+1}}$ converges (as a geometrical progression with the common ratio $\frac{|z - z_0|}{\rho} < 1$). Hence a termwise integration can be carried out:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \sum_0^\infty f(\zeta)\,\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\,d\zeta = \sum_0^\infty \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d'\zeta \cdot (z - z_0)^n.$$

The coefficients of the power series,

$$a_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \qquad (n=0, 1, 2, \ldots),$$

thus obtained do not depend on the radius ρ. In fact, if $\rho_1 \neq \rho$ and $0 < \rho_1 < r$, then by the theorem on a composite contour we have :

$$\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}} = \int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}} \ .$$

Since the expansion

$$f(z) = \sum_0^\infty a_n\,(z-z_0)^n$$

has been established for any arbitrary point z of the circle $|z-z_0| < r$, the theorem is proved.

We now put $\max\limits_{\zeta\in\gamma_\rho} |f(\zeta)| = M(\rho)$. From the formulae obtained above for the coefficients of the power series we obtain *the Cauchy inequalities*

$$|a_n| \leqslant \frac{1}{2\pi} \frac{M(\rho)}{\rho^{n+1}} \cdot 2\pi\rho = \frac{M(\rho)}{\rho^n} \qquad (n=0, 1, 2, \ldots). \tag{8}$$

These are used to estimate, from the upper bound, the moduli of the coefficients of the power series in terms of the maximum module of the sum of the series on the circle $|z - z_0| = \rho$ and its radius.

We get the following estimate for the remainder of the series :

$$\left|\sum_{n+1}^\infty a_k (z-z_0)^k\right| \leqslant \sum_{n+1}^\infty \frac{M(\rho)}{\rho^k} |z - z_0|^k = \frac{M(\rho)\,|z-z_0|^{n+1}}{\rho^n(\rho-|z-z_0|)}, \tag{9}$$

which gives us an idea of the magnitude of the error of the approximation

$$f(z) \approx \sum_0^n a_k\,(z-z_0)^k.$$

With the help of the formula (8) it is easy to prove Liouville's theorem, namely *that an entire function with a bounded module is a constant.*

Let $f(z)$ be an entire function. Then, its expansion into a power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

converges in the whole of the plane (here r, the distance of the point $z_0 = 0$ from the boundary of the finite plane i.e. from the point ∞, is

equal to ∞). If $f(z)$ is bounded in its module, then $|f(z)| \leqslant M$, where M is a positive constant. Then

$$|a_n| \leqslant \frac{M(\rho)}{\rho^n} \leqslant \frac{M}{\rho^n} \quad (n=0, 1, 2, \ldots),$$

where we can choose for ρ any positive number. Considering $n \geqslant 1$, we let ρ tend to ∞. Then, we obtain

$$|a_n| \leqslant 0, \quad \text{i.e.} \quad a_n = 0, \quad n = 1, 2, 3, \ldots .$$

Hence

$$f(z) \equiv a_0.$$

Thus, the theorem is proved.

As a simple example of the application of Liouville's theorem, we shall prove the so-called *fundamental theorem of higher algebra*, viz. *that any polynomial* $P(z) = c_0 + c_1 z + \ldots + c_n z^n$ $(n \geqslant 1, c_n \neq 0)$ *has at least one zero.*

We start from the reverse. Let $P(z)$ have no zero. Then $f(z) = \frac{1}{P(z)}$ is an entire function satisfying the condition $\lim\limits_{z \to \infty} f(z) = 0$. This function is bounded in its module in the whole plane (in fact, there exists an $R > 0$ such that, for $|z| > R$, $|f(z)| < 1$; if $\max\limits_{|z| \leqslant R} |f(z)| = m$, then $|f(z)| < m + 1$ for all z). Hence $f(z) \equiv$ const $= 0$, which contradicts the very definition of this function.

3. INFINITE DIFFERENTIABILITY OF ANALYTIC AND HARMONIC FUNCTIONS

Following are the important consequences of the results of Art. 2.

I. *Every function* $f(z)$ *which is analytic in the domain* G *has derivatives of all orders in this domain, that is, is infinitely differentiable in it.*

In fact, $f(z)$ can be decomposed into a power series in the neighbourhood of any point $z_0 \in G$ and hence is infinitely differentiable in this neighbourhood (*cf.* Art. 3, Chapter 4).

According to Art. 3 (Chapter 4) the decomposition of the function $f(z)$ can be written in the form

$$f(z) = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (|z - z_0| < r).$$

In the special case when $f(z)$ is an entire function, the distance r of the point z_0 from the boundary of the finite plane (i.e. from the point at

infinity) is equal to infinity. Putting $z_0 = 0$, we obtain

$$f(z) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} z^n, \quad |z| < \infty.$$

Calculating the derivatives of various orders of the elementary functions $\exp z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, the reader can easily obtain for them the following expansions that converge in the whole plane:

$$\exp z = \sum_0^\infty \frac{z^n}{n!}, \quad \sin z = \sum_1^\infty (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}, \quad \cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sinh z = \sum_1^\infty \frac{z^{2n-1}}{(2n-1)!}, \quad \cosh z = \sum_0^\infty \frac{z^{2n}}{(2n)!}.$$

We shall now discuss the power expansions of the one-valued branches of the simplest many-valued functions. First of all we examine the one-valued branch of $\ln z$ defined by the condition $\ln 1 = 0$ in the domain G having as its boundary the nonpositive part of the real axis: $x \leqslant 0$, $y = 0$ (this branch can be represented in the form $\ln z = \ln |z| + i \arg z$, where $|\arg z| < \pi$). Since $(\ln z)' = \frac{1}{z}$ so $(\ln z)^{(k)} = (-1)^{k-1} \frac{(k-1)!}{z^k}$, and, hence, the expansion of $\ln z$ into powers of $(z-1)$ assumes the form

$$\ln z = \sum_0^\infty \frac{(\ln z)^{(k)}_{z=1}}{k!} (z-1)^k$$

or

$$\ln z = \sum_1^\infty (-1)^{k-1} \frac{(z-1)^k}{k}.$$

The distance r of the point $z_0 = 1$ from the boundary of the domain G is equal to unity. Hence, for $|z-1| < 1$ we can use the expansion obtained above. Replacing $(z-1)$ by z here, we obtain the power series for $\ln(1+z)$ which converges for $|z| < 1$:

$$\ln(1+z) = \sum_1^\infty (-1)^{k-1} \frac{z^k}{k} \quad (|z|<1).$$

$\ln(1+z)$ is a one-valued branch of the function $\operatorname{Ln}(1+z)$ defined by the condition $[\operatorname{Ln}(1+z)]_{z=0} = \ln 1 = 0$, in the domain G having the part of the real axis $x \leqslant -1$, $y = 0$ as its boundary.

We further examine the function z^α, where α is an arbitrary complex number. This is a many-valued function (when α is not an integer); this is expressible in terms of an exponential function and a logarithm by the relation $z^\alpha = \exp(\alpha \operatorname{Ln} z)$.

We isolate a one-valued branch $\varphi(z)$ of this function, in the same domain G which we had in the previous example, through the condition $\varphi(1) = 1$. This branch may be represented in the form

$$\varphi(z) = \exp(\alpha \ln z)$$

whence

$$\varphi'(z) = \alpha \exp(\alpha \ln z) \cdot \frac{1}{z} = \alpha \exp(\alpha \ln z) \exp(-\ln z) =$$

$$= \alpha \exp[(\alpha-1) \ln z]$$

and, further

$$\varphi^{(k)}(z) = \alpha(\alpha-1) \ldots (\alpha-k+1) \exp[(\alpha-k) \ln z] \quad (k=1, 2, \ldots).$$

At the point $z = 1$ the derivatives $\varphi^{(k)}(1) = \alpha(\alpha-1) \ldots (\alpha-k+1)$ and consequently, the expansion of $\varphi(z)$ in powers of $(z-1)$ assumes the form

$$\varphi(z) = 1 + \sum_1^\infty \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!} (z-1)^k.$$

This is true, as in the previous case, for $|z-1| < 1$. Replacing $z-1$ by z, we again obtain an expansion of $\varphi(z+1)$ into a power series in z. Using the usual notation $(1+z)^\alpha$ instead of $\varphi(z+1)$ [$(1+z)^\alpha$ understood to represent the function $\exp[\alpha \ln (1+z)]$, one-valued in the domain G], we obtain

$$(1+z)^\alpha = 1 + \sum_1^\infty \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!} z^k \quad (|z|<1).$$

This is the *binomial formula* established for here the most general case when the exponent is an arbitrary complex number.

II. *The derivative of an analytic function is itself an analytic function.* In fact, $f'(z)$ possesses the derivative $[f'(z)]' = f''(z)$ everywhere in the domain G.

III. *The real and the imaginary parts of an analytic function possess differentiable partial derivatives of all orders.*

In fact, if $f(z) = u(x, y) + iv(x, y)$ is a function analytic in the domain G, then by virtue of Art. 7, Chapter 2 $u(x, y)$ and $v(x, y)$ are

differentiable functions of x and y in this domain. Since

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

hence, it follows that the partial derivatives of the first order of the functions u and v are also differentiable in this domain. To prove the proposition III it is sufficient to show that for any nonnegative integers k and l the partial derivative $\frac{\partial^{k+l}u}{\partial x^k \partial y^l}\left(\text{or } \frac{\partial^{k+l}v}{\partial x^k \partial y^l}\right)$ exists and is the real or imaginary part of a certain analytic function [viz. of the function $\pm f^{(k+l)}(z)$].

Let this statement be true for all derivatives of order n, so that we have, for example,

$$\frac{\partial^{k+l}u}{\partial x^k \partial y^l} = \operatorname{Re} F(z) = U(x, y), \text{ where } k + l = n.$$

Then there exist

$$\frac{\partial U}{\partial x} = \frac{\partial^{k+l+1}u}{\partial x^{k+1}\partial y^l} = \operatorname{Re} F'(z) \text{ and } \frac{\partial U}{\partial y} = \frac{\partial^{k+l+1}u}{\partial x^k \partial y^{l+1}} = -\operatorname{Im} F'(z),$$

whence follows the validity of the statement for partial derivatives of order $n + 1$ and, consequently, also the validity of the proposition III.

In particular, there exist continuous (by virtue of their differentiability) partial derivatives, of second order, of the functions $u(x, y)$ and $v(x, y)$. We made use of exactly this fact in Art. 12 of Chapter 2, where it was proved that the real and the imaginary parts of an analytic function are harmonic functions (conjugate to each other).

IV. *If L is a closed rectifiable Jordan curve with its interior D lying in the domain G, then at every point $z \in D$, for any natural number k, the equality*

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{k+1}}. \tag{10}$$

holds.

In fact, if γ_r be a circle, with its centre at z lying in the interior of L, then

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{k+1}} = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{k+1}} = a_k,$$

where a_k is the coefficient of $(z' - z)^k$ in the power series representing $f(z')$ in the neighbourhood of z. But this coefficient is also expressible

through a derivative, of order k, of the sum of the series at the point z by means of the formula

$$a_k = \frac{f^{(k)}(z)}{k!}.$$

The proposition IV follows from a comparison of the last two formulae.

We note that the equality proved above may be formally obtained from the Cauchy formula $f(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{\zeta - z}$, if we differentiate on both sides k times with respect to z. Differentiation on the right-hand side is performed under the integral sign.

Thus, the proposition proved establishes indirectly the validity of such a differentiation. We may, however, also prove directly the validity of differentiation under the integral sign under the given condition.

4. THE THEOREM OF MORERA

Every function $f(z)$, analytic in the simply-connected domain G, is continuous in this domain and possesses the property that its integral along any curve depends only on the initial and final points of the path of integration. Evidently, this proposition differs from the Cauchy integral theorem only in the context of its formulation. The advantage of such a formulation of the Cauchy theorem lies in that it enables us to obtain the converse theorem:

The theorem of Morera. *If the function $f(z)$ is continuous in the simply-connected domain G and its integral along any curve depends only on the initial and final points of the path of integration, then this function is analytic in the domain G.*

For proving this we note that the integral $\int_{z_0}^{z} f(z)\,dz$, where z_0 is fixed and z is a variable point of the domain, represents, in the given case, a one-valued function $F(z)$. In Art. 7 of Chapter 5, it was proved, under the same assumptions as those of the given theorem, that $F(z)$ is an analytic function and that $F'(z) = f(z)$, i.e. $f(z)$ is the derivative of an analytic function. Then, by the proposition II of Art. 3, it follows that $f(z)$ is an analytic function.

The conditions of the theorem of Morera can be somewhat relaxed. First of all, instead of assuming that the integral of $f(z)$ for any curve depends only on the initial and final points of this curve, it is sufficient to stipulate that this integral vanishes along any triangular contour lying in

the domain G. In fact, it follows from this assumption (as it was established while proving the Cauchy integral theorem) that the integral would also vanish along any closed polygon and for that matter, any closed rectifiable contour. But, this last fact is tantamount to saying that $\int_L f(z)\,dz$ depends only on the initial and final points of L where L is any curve. Further, for the sake of a theorem which will be proved later (Art. 5, Chapter 9) we note here that instead of the condition that the integral along any triangle vanishes we could only require that it does so along triangular contours not having points in common with a given segment δ of some straight line R.

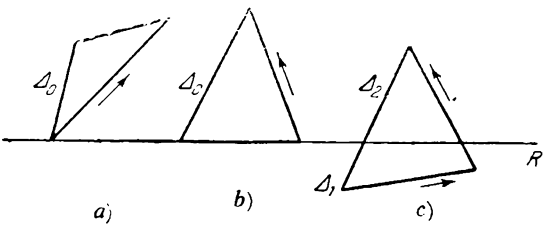


Fig. 46.

Indeed, if the triangular contour $\Delta_0 \subset G$ has points in common with δ, then it is located either on one side of R [cases (a) and (b) in Fig. 46] or is divided by R into two convex polygons Δ_1 and Δ_2 [case (c) in Fig. 46]. In each of these cases the function $f(z)$ is continuous on the contour Δ_j ($f=0, 1, 2$) and in its interior and besides, it is also analytic in the interior of Δ_j (because the theorem of Morera is applicable to $f(z)$ in the interior of Δ_j). It has been shown in Art. 8 of Chapter 5 that, under these conditions, $\int_{\Delta_j} f(z)\,dz = 0$. In cases (a) and (b), we, consequently, get $\int_{\Delta_0} f(z)\,dz = 0$; in case (c) $\int_{\Delta_1} f(z)\,dz = 0$ and $\int_{\Delta_2} f(z)\,dz = 0$, whence adding up we get: $\int_{\Delta_0} f(z)\,dz = 0$. Thus, we have verified that the integral vanishes also along triangles intersecting R. Consequently, $f(z)$ is analytic in the whole of the domain G (including the points of R).

5. WEIERSTRASS'S THEOREM ON THE UNIFORMLY CONVERGENT SERIES OF ANALYTIC FUNCTIONS

Let the terms of the series $\sum_1^\infty f_n(z)$ *be one-valued and analytic functions in the domain* G *and let this series converge uniformly in every*

closed circle contained in this domain. Then : a) the sum of the series $f(z)$ *is analytic in this domain, b) the series can be differentiated termwise any number of times :* $\sum_1^\infty f_n^{(k)}(z) = f^k(z)$, *and c) all the differentiated series converge uniformly in any closed circle contained in domain G.*

In the sequel, for the sake of brevity, a series uniformly convergent in any closed circle belonging to some domain G, will be called *uniformly convergent in the interior of the domain G.* Using the Hiene-Borel theorem on covering it can be easily proved *that uniform covergence in the interior of the domain G is equivalent to uniform convergence on any bounded closed set of points belonging to this domain.* We note that for uniform convergence in the interior of a domain G we do not require that the series be uniformly convergent in the entire domain G. Hence, the theorem is applicable, for example, to any power series in the region of its circle of convergence (*cf.* Art. 4, Chapter 4).

For the validity of the theorem it is essential that its conditions are satisfied in the *domain of the plane*, i.e. for any point in G there exists a circle, with its centre at this point, contained in the set G. If, for example, we examine the series of the simplest of analytic functions, viz., the polynomial $P_n(z)$ uniformly convergent on a segment of the real axis (on a set none of whose points satisfies just the above condition), then the sum of such a series is either not differentiable on this segment or, if it is differentiable, it is not always termwise differentiable.[1] Thus, the theorem of Weierstrass expresses a specific property of analytic functions of complex variable.

In order to prove this we first note that it is sufficient to prove the theorem for any arbitrary circle K : $|z-z_0| < \rho$ belonging, along with its boundary, to the given domain. To prove the statement *a)* we shall apply the theorem of Morera. Since the series uniformly converges in $\bar{K}$ and its terms are continuous, hence its sum $f(z)$ is also continuous in K and it can be integrated termwise along any triangular contour Δ contained in K. Hence

$$\int_\Delta f(z)\,dz = \sum_1^\infty \int_\Delta f_n(z)\,dz = 0$$

(we have used here the analyticity of the functions $f_n(z)$ in the circle K). Thus it follows from Morera's theorem that $f(z)$ is a function analytic

[1]See, in this connection, G. M. Fichtenholtz, loc. cit., Vol. II, 3rd Ed., Gostekhizdat, p. 461.

in K, and, consequently, also at all points of the domain G. Further, if $z \in K$, then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int\limits_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

(*cf.* proposition IV of Art. 3). But $\frac{f(\zeta)}{(\zeta - z)^{k+1}} = \sum\limits_{n=1}^{\infty} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}}$, where for a fixed z the last series converges uniformly on the circle $|\zeta - z_0| = \rho$ (it is obtained from the uniformly convergent series $\sum\limits_{n=1}^{\infty} f_n(\zeta)$ on multiplying it by a function with a bounded module $\frac{1}{|\zeta - z|^{k+1}} \leqslant \frac{1}{(\rho - |z - z_0|)^{k+1}}$). Hence this series can be integrated termwise with respect to ζ. Consequently,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int\limits_{|\zeta - z| = z = \rho} \sum_{n=1}^{\infty} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta =$$

$$= \sum_{n=1}^{\infty} \frac{k!}{2\pi i} \int\limits_{|\zeta - z_0| = \rho} \frac{f_n(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} = \sum_{n=1}^{\infty} f_n^{(k)}(z),$$

i.e. the statement (*b*) is true.

Finally, to prove (*c*) we take the circle $K' : |z - z_0| < \rho'$ where $\rho' > \rho$ but less than the distance of z_0 from the boundary of the domain G. The closed circle $\overline{K'}$ is contained in G and contains the closed circle $\overline{K}$ in its interior. For every $z \in \overline{K}$, we have

$$\left| f^{(k)}(z) - \sum_{1}^{N} f_n^k(z) \right| \left| \frac{k!}{2\pi i} \int\limits_{|\zeta - z_0| = \rho'} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \right.$$

$$\left. - \sum_{1}^{N} \frac{k!}{2\pi i} \int\limits_{|\zeta - z| = \rho'} \frac{f_n(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \right| \leqslant \frac{k!}{2\pi} \int_{0}^{2\pi} \frac{\left| f(\zeta) - \sum\limits_{1}^{N} f_n(\zeta) \right|}{|\zeta - z|^{k+1}} \rho'\, d\varphi <$$

$$< \frac{k!}{2\pi} \frac{2\pi\rho'}{(\rho' - \rho)^{k+1}} \sup_{|\zeta - z_0| = \rho'} \left| f(\zeta) - \sum_{1}^{N} f_n(\zeta) \right|.$$

By virtue of the uniform convergence of the series $\sum_1^\infty f_n(\zeta)$ on the circle $|\zeta - z_0| = \rho$ the last factor may be made as small as desired if N is taken to be sufficiently large (irrespective of the position of the point $z \in \overline{K}$). Thus, for any $\varepsilon > 0$ and $z \in \overline{K}$

$$\left| f^{(k)}(z) - \sum_1^N f_n^{(k)}(z) \right| < \varepsilon, \quad \text{if } N > N_0(\varepsilon).$$

Hence, Weierstrass's theorem is completely proved.

It was shown in Art. 2 of this chapter that the function $f(z)$, analytic in the circle $|z - z_0| < R$, can be decomposed into a Taylor series in this circle. Since the power series converges uniformly in a concentric circle with a smaller radius $|z - z_0| \leqslant r$ $(r < R)$, it follows that in every such circle, for any $\varepsilon > 0$, the inequality

$$\left| f(z) - \sum_0^n \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right| < \varepsilon, \quad n > N(\varepsilon, r)$$

is satisfied. In other words, *there exists a polynomial which approximates to a function $f(z)$, analytic in the circle $|z - z_0| \leqslant r$, with any desired degree of accuracy.* We can now prove the converse : *If the function $f(z)$, defined in the circle $|z - z_0| < R$, can be approximated by a polynomial to any desired degree of accuracy in every concentric circle with a smaller radius, this function is analytic in this circle.*

To prove this, let $\{r_n\}$ be an increasing sequence of positive numbers, converging to R; for every circle $|z - z_0| \leqslant r_n$ we construct a polynomial $p_n(z)$ such that $|f(z) - p_n(z)| \leqslant \frac{1}{n}$ for $|z - z_0| \leqslant r_n$. Then, as can be easily seen, $f(z) = p_1(z) + \sum_{k=2}^\infty [p_k(z) - p_{k-1}(z)]$, i.e. $f(z)$ can be represented, in the circle $|z - z_0| < R$, as a series of polynomials uniformly convergent in any smaller circle. Hence, by Weierstrass's theorem, it follows that $f(z)$ is analytic in the circle $|z - z_0| < R$.

In view of the foregoing, we have the following proposition.

In order that the function $f(z)$ be analytic in the circle $|z - z_0| < R$, it is necessary and sufficient that it can be approximated, to any desired degree of accuracy, by a polynomial in every circle concentric to the above and of a smaller radius.

Thus the property (d) of Art. 2 of the Introduction is established.

6. THE UNIQUENESS THEOREM

One of the most important properties of analytic functions is expressed by the *uniqueness theorem.* It states that *there can exist at most one function $f(z)$ which is one-valued and analytic in the domain G, takes up given values on any set of points E in this domain and possesses at least one finite limit point $z_0 \in G$.*

Before proving the uniqueness theorem we shall explain it by some examples.

a) Let G be a finite plane and E the interval $0 < x < \frac{\pi}{2}$ on the real axis. Obviously, every point of E in this case is a limit point for E and lies in G. By the theorem there can exist at most one function $f(z)$, analytic in the finite plane, the value of which for $z = x \in E$ coincides, say, with $\sin x$: $f(x) = \sin x \left(0<x<\frac{\pi}{2}\right)$.

We know that such an analytic function does actually exist. It is $f(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sin z$. The uniqueness theorem asserts that this is the only function that coincides with $\sin x$ at the points of E. A similar assertion also holds for the function $\varphi(z) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$.

b) The uniqueness theorem implies that certain relations established between analytic functions, under special restrictions with respect to the values of the argument, are also valid without such restrictions. For example, the validity of the relation $\sin^2 z + \cos^2 z = 1$ in the interval $0 < z = x < \frac{\pi}{2}$ follows directly from the theorem of Pythagoras. To establish its validity for any complex z it is enough to point out that the function $F(z) = \sin^2 z + \cos^2 z$ which is analytic in the whole of the plane, takes up on E a value equal to unity. But the analytic function $\Phi(z) \equiv 1$ takes up the same values on E. Thus, by the uniqueness theorem $F(z)$ and $\Phi(z)$ must be one and the same analytic function, i.e. $\sin^2 z + \cos^2 z \equiv 1$ in the whole of the plane.

c) We now investigate if there exists a function, analytic in the whole of the finite plane G, satisfying the conditions $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$. Let E denote the set of points, $1, \frac{1}{2}, \frac{1}{3}, \ldots, \ldots, \frac{1}{n}, \ldots$. It is obvious that this set has a limit point 0 lying in G. Since the analytic function $F(z) = z$ takes up at the points of E given values of $F\left(\frac{1}{n}\right) = \frac{1}{n}$ $(n=1, 2, 3, \ldots)$, hence there does not exist any other function satisfying these conditions and consequently, $f(z) \equiv F(z) \equiv z$.

But $f(z) = z$, at the points $-\frac{1}{n}$, takes up the values $-\frac{1}{n}$, and not $\frac{1}{n}$, as was required by the condition for the problem. Hence, we encounter complete negation to the problem. Actually, no function, analytic in G, can satisfy the conditions

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n} \quad (n=1, 2, 3, \ldots).$$

d) Finally, suppose that we have to find the function $f(z)$, analytic in the finite plane, given that $f(k\pi) = 0$ $(k=0, \pm 1, \pm 2, \ldots)$. Here, the values of the function are given on the set of points E: $0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots$, and this set has no limit point in the finite plane. Hence, we might apply the uniqueness theorem formulated above. It can be easily seen that in the given case it does not hold. In fact, there exists an infinite set of different analytic functions satisfying the stipulated conditions viz. $f(z) \equiv 0$, $\sin z$, $C \sin z$, $\sin^2 z$, $\sin^3 z, \ldots, \sin^n z, \ldots$.

We now turn to prove the theorem. Let us assume that $f(z)$ and $\varphi(z)$ are two analytic functions taking one and the same value at all points of E. Then their difference $\psi(z) = f(z) - \varphi(z)$ is a function analytic in G and vanishes at every point of E. If we prove that $\psi(z) \equiv 0$, $z \in G$, then it would follow that $f(z) \equiv \varphi(z)$. Thereby, the theorem would be proved.

First we examine the special case when G is a circle with a finite or an infinite radius (in the latter case, G coincides with the whole plane) with its centre at the point z_0, the limit point of E. The function $\psi(z)$, analytic in G and vanishing on E, can be represented by a power series convergent in the whole circle :

$$\psi(z) = c_0 + c_1(z-z_0) + \ldots + c_n(z-z_0)^n + \ldots . \qquad (11)$$

We shall show that all the coefficients of the series are zero. From this it would follow that $\psi(z) \equiv 0$.

Since z_0 is the limit point of the set E, we can find a sequence of different points $z_1, z_2, \ldots, z_k, \ldots$, distinct from z_0, such that it converges to z_0.

Thus, we have

$$\psi(z_k) = c_0 + c_1(z_k-z_0) + \ldots + c_n(z_k-z_0)^n + \ldots = 0 \qquad (12)$$

$(k=1, 2, 3 \ldots)$; passing over to the limit $z_k \to z_0$, we obtain

$$\psi(z_0) = c_0 = 0.$$

Suppose it has already been proved that the coefficients $c_0, c_1, \ldots$ $\ldots, c_{n-1}$ are zero ; we get

$$\psi(z_k) = c_n (z_k - z_0)^n + c_{n+1} (z_k - z_0)^{n+1} + \ldots = 0,$$

or

$$c_n + c_{n+1} (z_k - z_0) + \ldots = 0 \quad (k = 1, 2, \ldots). \tag{13}$$

Passing over to the limit $z_k \to z_0$, in the relations (13), we obtain

$$c_n = 0.$$

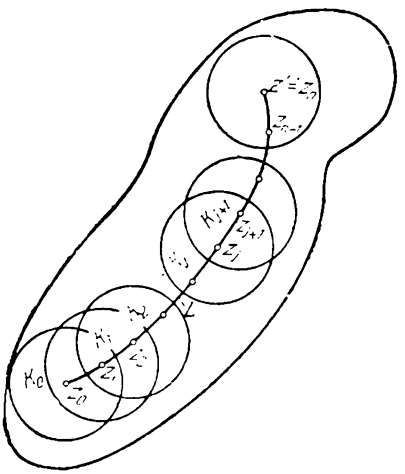


Fig. 47.

Thus, all the coefficients of the series (11) vanish, whence it follows that the theorem is valid when the domain is a circle and its centre is the limit point of E. We now examine the general case. Let K be any circle contained in the domain G with its centre at the point z_0, the limit point of E. As proved above, the function $\psi(z)$, vanishing at every point of E, lying in K, vanishes in the entire circle K. Since G does not coincide with this circle, there must exist points in the domain G which are not contained in K. We shall show that $\psi(z)$ vanishes at any such point z'. We connect z_0 and z' in the interior of G by a continuous curve L and let $\rho > 0$ be the distance between L and the boundary Γ of the domain G. Let us divide L into arcs by the points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n = z'$ such that the distance between any two neighbouring points is less than ρ and draw a circle K_j of radius ρ (Fig. 47) with every z_j as a centre. Evidently, the interior of K_j lies in G and contains the centre z_{j+1} of the next circle ($j = 0, 1, \ldots, n-1$). The function $\psi(z)$ vanishes at all points of the circle K_0. In general, we assume it proved that $\psi(z) = 0$ at all points of the circles K_j ($j \leqslant n-1$). Now we shall prove that $\psi(z) = 0$ at all points of the circle K_{j+1}.

In fact, the centre z_{j+1} of the circle K_{j+1} lies in the interior of K_j and hence, is a limit point of the set on which $\psi(z) = 0$. According to the special case of the theorem proved above, it follows that $\psi(z) = 0$ in K_{j+1}. Hence, it follows that $\psi(z) = 0$ also in K_n and particularly at the centre $z_n = z'$ of this circle. Thus, $\psi(z) \equiv 0$, $z \in G$. This completes the **proof of the theorem.**

7. *A*-POINTS AND, IN PARTICULAR, THE ZEROS

Let A be an arbitrary finite complex number. We shall call the roots of the equation $f(z) = A$, *A-points* of the function $f(z)$ which is analytic in a certain domain G. It follows from the uniqueness theorem that when $f(z) \not\equiv A$ the set of A-points cannot have any limit point belonging to the domain G (assuming the contrary, we would have that $f(z) \equiv A$). Hence, it follows, in particular, *that any bounded closed set F of the domain G can contain only a finite number of A-points (for any fixed A).*

Now, assuming that F contains an infinite set of A-points we find that this set has a limit point belonging to $\bar{F}$ and, hence, to F.

Let z_0 be any A-point of the function $f(z)$ such that $f(z_0) = A$. In the neighbourhood of the point z_0, we have

$$f(z) = A + f'(z_0)(z-z_0) + \frac{f'(z_0)}{2!}(z-z_0)^2 + \ldots,$$

or

$$f(z) - A = f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \ldots . \qquad (14)$$

If $f(z) \not\equiv A$, we find nonzero coefficients among the coefficients on the right-hand side here. Let $(z-z_0)^k$ be the least power of $z-z_0$ the coefficient of which is different from zero. Then, equation (14) takes the form

$$f(z) - A = (z-z_0)^k \left[\frac{f^{(k)}(z_0)}{k!} + \frac{f^{(k+1)}(z_0)}{(k+1)!}(z-z_0) + \ldots\right], \qquad (15)$$

where $f^{(k)}(z_0) \neq 0$. The natural number k is called the *order* or the *multiplicity* of the A-point z_0

When $k = 1$ the A-point is called *simple* and when $k > 1$, it is called *multiple.*

By definition, a simple point is characterized by the fact that for it $f(z_0) = A$ and $f'(z_0) \neq 0$. A multiple A-point of order $k \geqslant 2$ is characterized by the relations

$$f(z_0) = A,\ f'(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0,\ f^{(k)}(z_0) \neq 0.$$

All that has been said above can be extended, in particular, to the case when $A = 0$. 0-points of an analytic function are briefly called it *zeros,* It can be easily verified that the definitions of multiplicity of a zero of a polynomial or in general of any rational function adopted in algebra agree with the general definitions given above. We note that for every $A \neq 0$ the A-points of the function $f(z)$ are zeros of the function $f(z) - A$.

Examples

(*a*) For $\sin z$ all its zeros $z = k\pi$ ($k=0, \pm 1, \pm 2, \ldots$) are simple because $(\sin z)' = \cos z$ does not vanish for $z = k\pi$.

(*b*) For the function $\cos z$, the 1-points are $z = 2k\pi$ ($k=0, \pm 1, \pm 2, \ldots$). All these have a multiplicity 2, because $(\cos z)' = -\sin z = 0$ for $z = 2k\pi$, whereas $(\cos z)'' = -\cos z \neq 0$ for $z = 2k\pi$. Hence, it follows that all the zeros of the function $\cos z - 1$ (or $1-\cos z$) are double, i.e. they have a multiplicity 2.

(*c*) For the function $f(z) = \sin z - z$, the point $z = 0$ is a zero with multiplicity 3 because $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = -\cos 0 \neq 0$.

8. SERIES OF POWER SERIES

We now consider some examples of the expansion of analytic functions into power series. In principle, the question of finding a Taylor's expansion is solved by finding formulae for the calculation of coefficients of the series

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n=0, 1, 2, \ldots).$$

But direct calculations based on the computation of successive derivatives of the function $f(z)$ are often found to be very unwieldly or laborious. However, in several practically important cases, Taylor's expansions can be realized deducing their definite images from other earlier well known expansions.

We assume that $f(z)$ can be represented in the form of a series of analytic functions uniformly convergent in the interior of a certain neighbourhood $|z-z_0| < \rho$ of the point z_0:

$$f(z) = \sum_1^\infty f_n(z). \tag{16}$$

Then, by Weierstrass's theorem, we will have

$$f^{(k)}(z) = \sum_1^\infty f_n^{(k)}(z)$$

and

$$\frac{1}{k!} f^{(k)}(z_0) = \sum_1^\infty \frac{f_n^{(k)}(z_0)}{k!}. \tag{17}$$

Here $\frac{1}{k!} f_n^{(k)}(z_0)$ represents the coefficient of $(z-z_0)^k$ in the Taylor expansion of the function $f_n(z)$ and $\frac{1}{k!} f^{(k)}(z_0)$ the coefficient of $(z-z_0)^k$ in the Taylor expansion of $f(z)$.

Hence, *the Taylor coefficients of the sum of a uniformly convergent series of analytic functions* $\sum_1^\infty f_n(z)$ *are obtained by adding together the like Taylor coefficients* [*i.e. the coefficients of the same power* $(z-z_0)^k$], *taken from expansion of all the functions* $f_n(z)$.

We shall give here two examples. First, we consider the sum of the series

$$F(z) = \sum_1^\infty \frac{z^n}{1-z^n}.$$

The terms of this series are functions of z, analytic in the unit circle; moreover, the series $\sum_1^\infty \frac{z^n}{1-z^n}$ converges uniformly in the interior of this circle. In fact, if E is a closed set of points of this circle and $\delta > 0$ is the distance of E to the unit circle, then for any point $z \in E$, we have: $|z| \leqslant 1 - \delta = \rho < 1$. Hence

$$\left|\frac{z^n}{1-z^n}\right| \leqslant \frac{\rho^n}{1-\rho^n} \leqslant \frac{\rho^n}{1-\rho},$$

and since the series $\sum_1^\infty \frac{\rho^n}{1-\rho}$ converges (it is a geometric progression with common ratio ρ), hence the given series converges uniformly on E, i.e. it converges uniformly in the interior of the unit circle. To find the coefficient of z^k in the Taylor expansion of the function $F(z)$, it is necessary to add up the coefficients of z^k in the Taylor expansions of the functions

$$\frac{z^n}{1-z^n} = z^n + z^{2n} + z^{3n} + \ldots \quad (n=1, 2, \ldots).$$

The coefficient of z^k in such an expansion is equal to zero, if k is not divisible by n; it is equal to unity, if k is divisible by n. Consequently, the coefficient of z^k in the expansion of $F(z)$ is equal to the sum of unities whose number is equal to the number of all natural factors of the number k. Denoting it by $\tau(k)$ [$\tau(1) = 1$, $\tau(2) = 2$, $\tau(3) = 2$, $\tau(4) = 3$,

$\tau(5) = 2, \ldots$], we have

$$F(z) = \sum_{1}^{\infty} \tau(k)\, z^k.$$

This is the expansion sought for. Since $F(z)$ is a function analytic in the unit circle (by Weierstrass's theorem), the expansion obtained converges in the unit circle.

We further note that when $z = 1$ the series diverges, because it assumes the form $\sum_{1}^{\infty} \tau(k)$ in which all the terms are natural numbers. It thus follows that the radius of convergence of the series is equal to unity.

We take another example, of the series

$$\Phi(z) = \sum_{1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}.$$

The function $\frac{2z}{z^2-n^2\pi^2}$ is analytic everywhere except at the points $z = \pm n\pi$, where it becomes ∞. Consequently, each of these functions is analytic in the circle $|z| < \pi$. We shall show that the given series converges uniformly in the interior of this circle. In fact, if $|z| \leqslant \rho$, where $\rho < \pi$, then

$$\left|\frac{2z}{z^2-n^2\pi^2}\right| \leqslant \frac{2\rho}{n^2\pi^2-\rho^2} = \frac{2\rho}{n^2\pi^2} \cdot \frac{1}{1-\frac{\rho^2}{n^2\pi^2}} \leqslant \frac{2\rho}{n^2\pi^2} \cdot \frac{1}{1-\frac{\rho^2}{\pi^2}},$$

and since on the right-hand side we obtain a general term of a convergent series $\left(\text{series } \sum_{1}^{\infty} \frac{1}{n^2} \text{ converges}\right)$, hence the series $\sum_{1}^{\infty} \frac{2z}{z^2-k^2\pi^2}$ converges uniformly in the interior of the circle $|z| < \pi$. Consequently, the coefficient of z^k in the Taylor expansion of the function $\Phi(z)$ is equal to the sum of the coefficients of the same power of z in the Taylor expansions of all the functions $\frac{2z}{z^2-n^2\pi^2}$. This expansion has the form

$$\frac{2z}{z^2-n^2\pi^2} = -\frac{2z}{n^2\pi^2} \cdot \frac{1}{1-\left(\frac{z}{n\pi}\right)^2} =$$

$$= -\frac{2z}{n^2\pi^2} - \frac{2z^3}{n^4\pi^4} - \frac{2z^5}{n^6\pi^6} - \ldots - \frac{2z^{2q-1}}{n^{2q}\pi^{2q}} - \ldots;$$

here, the coefficient of z^k is equal to zero if k is even and is equal to $-\frac{2}{n^{2q}\pi^{2q}}$ if $k = 2q - 1$ (odd). Hence, the coefficient of z^k in the expansion of the function $\Phi(z)$ is equal to zero if k is an even number and equal to $-2\sum_{1}^{\infty}\frac{1}{(n\pi)^{2q}}$ if k is an odd number : $k = 2q - 1$.

Thus

$$\Phi(z) = -2\sum_{q=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{(\pi n)^{2q}}\right] z^{2q-1}.$$

This is the required power series. It must converge in the circle $|z| < \pi$, because the function $\Phi(z)$ is analytic in this circle. At the point $z = \pi$ we have the inequality

$$\sum_{n=1}^{\infty}\frac{1}{(\pi n)^{2q}}\pi^{2q-1} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^{2q}} > \frac{1}{\pi}$$

for each term of the series.

It thereby follows that the condition necessary for the convergence of the series is not fulfilled ; hence the series diverges. Thus, the radius of convergence of the series thus obtained is equal to π.

9. SUBSTITUTION OF ONE SERIES IN ANOTHER

Let $f(z)$ be written in the form

$$f(z) = F[\varphi(z)], \tag{18}$$

in which

$$\varphi(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_n z^n + \ldots \quad (|z| < r) \tag{19}$$

and

$$F(w) = A_0 + A_1(w-\alpha_0) + A_2(w-\alpha_0)^2 + \ldots + A_m(w-\alpha_0)^m + \ldots \tag{20}$$

$$(|w-\alpha_0| < R),$$

the coefficients of the power series for $\varphi(z)$ and $F(w)$ being known. Since, under the given situation, $\varphi(z) \to \alpha_0$ as $z \to 0$, we can find such a number ρ $(0 < \rho \leqslant r)$ that the module $|\varphi(z) - \alpha_0|$ will be less than R for $|z| < \rho$. Under this condition, the point $w = \varphi(z)$ belongs to the circle of convergence of the series (20) and, consequently, the function $f(z) = F(w) = F[\varphi(z)]$ is analytic for $|z| < \rho$. It thus follows that

there must exist an expansion of the function $f(z)$ into a power series in z, convergent for $|z| < \rho$. The problem then reduces to the calculation of the coefficients of this series.

We now examine the expansion

$$f(z) = F[\varphi(z)] = \sum_0^\infty A_n [\varphi(z) - \alpha_0]^n, \tag{21}$$

which, as we already know, is convergent for $|z| < \rho$. In order that we may refer to the uniform convergence of this series, we replace ρ by a number ρ' not greater than ρ $(0 < \rho' \leqslant \rho)$ such that the inequality

$$|\varphi(z) - \alpha_0| < \frac{R}{2}$$

is satisfied in the circle $|z| < \rho'$.

Since the series (20) converges uniformly for $|w - \alpha_0| < \frac{R}{2}$, the series (21) also converges uniformly for $|z| < \rho'$. Hence, the coefficient of z^k in the Taylor expansion of the function $f(z)$ can be calculated by taking the sum of the corresponding coefficients in the expansion of each of the functions $A_n [\varphi(z) - \alpha_0]^n$.

The latter expansions are obtained by an n-fold multiplication of the series for $\varphi(z) - \alpha_0$ by itself. In the given case, a term-by-term multiplication of the series is justified because we have here a power series in the circle of its convergence within which it converges absolutely.

Ultimately, we arrive at the following proposition. *In order to obtain a Taylor expansion of the function* $f(z) = F[\varphi(z)]$, *in which* $\varphi(z)$ *is a function analytic in the neighbourhood of the origin of coordinates and* $F(w)$ *is a function analytic in the neighbourhood of the point* $\alpha_0 = \varphi(0)$, *it is necessary to substitute the series* (19) *for* $w = \varphi(z)$ *in the series* (20), *for* $F(w)$ *perform the necessary involution, i.e. the multiplication of series, and, finally, add up the coefficients of terms having the same powers of* z. *The series thus obtained will give the required Taylor expansion of the function* $f(z)$. *It will surely converge in the circle* $|z| < \rho$, *where* ρ *is so chosen that for* $|z| < \rho$ *we have* $|\varphi(z) - \alpha_0| < R$.

The method described above for obtaining power expansions is called the method of substitution of *one series in another.*

We shall explain this by two examples.

Let

$$f(z) = \sqrt{\cos z}.$$

Of this two-valued function we examine that branch which is one-valued in the neighbourhood of the origin of coordinates and takes the value 1 for $z = 0$. In order to apply the rule under reference to this example, we express $f(z)$ in the form

$$f(z) = [1-(1-\cos z)]^{\frac{1}{2}}.$$

In our case

$$w = \varphi(z) = 1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots \quad (|z|<\infty),$$

and

$$F(w) = (1-w)^{\frac{1}{2}} = 1 - \frac{1}{2}w - \frac{1}{8}w^2 - \frac{1}{16}w^3 - \frac{5}{128}w^4 - \ldots \quad (|w|<1).$$

Hence

$$f(z) = F[\varphi(z)] = 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} - \ldots\right) - \\ - \frac{1}{8}\left(\frac{z^2}{2} - \frac{z^4}{24} + \ldots\right)^2 - \frac{1}{16}\left(\frac{z^2}{2} - \ldots\right)^3 - \ldots.$$

We confine ourselves to the calculation of the coefficients of powers from the first to the sixth inclusive. Then, we can evidently ignore the terms which are actually left unwritten because they would contribute to powers of z higher than the sixth. We thus have

$$\left(\frac{z^2}{2} - \frac{z^4}{24} + \ldots\right)^2 = \frac{z^4}{4} - \frac{z^6}{24} + \ldots, \quad \left(\frac{z^2}{2} - \ldots\right)^3 = \frac{z^6}{8} - \ldots,$$

and, hence,

$$f(z) = 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} - \ldots\right) - \frac{1}{8}\left(\frac{z^4}{4} - \frac{z^6}{24} + \ldots\right) - \\ - \frac{1}{16}\left(\frac{z^6}{8} - \ldots\right) + \ldots = 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19z^6}{5760} - \ldots.$$

Since $f(z) = \sqrt{\cos z}$ is a function analytic in the circle $|z| < \frac{\pi}{2}$ $\left(\text{having the derivative } -\frac{\sin z}{2\sqrt{\cos z}} \text{ in this circle}\right)$, the series for $f(z)$ must converge for $|z| < \frac{\pi}{2}$.

Next, we examine the function

$$f(z) = \exp\frac{1}{1-z}.$$

Writing $f(z)$ in the form $f(z) = e \exp \frac{z}{1-z}$, we put

$$w = \varphi(z) = \frac{z}{1-z} = z + z^2 + z^3 + \ldots \quad (|z|<1)$$

and

$$F(w) = e^{w+1} = e\left(1+\frac{w}{1!}+\frac{w^2}{2!}+\ldots\right) \quad (|w|<\infty).$$

Substitution of one series in the other gives

$$f(z) = e\left[1+(z+z^2+z^3+\ldots)+\frac{(z+z^2+z^3+\ldots)^2}{2!}+\right.$$
$$\left.+\frac{(z+z^2+z^3+\ldots)^3}{3!}+\ldots\right].$$

In this example, where $\varphi(z) = \frac{z}{1-z}$, we can avoid direct multiplication of the series, observing that for $|z|<1$

$$(z+z^2+z^3+\ldots)^k = \left(\frac{z}{1-z}\right)^k = z^k(1-z)^{-k} =$$
$$= z^k\cdot\left[1+\frac{k}{1}z+\frac{k(k+1)}{2!}z^2+\ldots+\frac{k\ldots(k+n-1)}{n!}z^n+\ldots\right],$$

and since

$$\frac{k(k+1)\ldots(k+n-1)}{n!} = \frac{(k+n-1)(k+n-2)\ldots(n+1)}{(k-1)!} = \binom{k+n-1}{k-1},$$

we have

$$(z+z^2+z^3+\ldots)^k = \sum_{n=0}^{\infty}\binom{k+n-1}{k-1}z^{n+k}.$$

Therefore

$$f(z) = e\left[1+\frac{1}{1!}\sum_0^\infty z^{n+1}+\frac{1}{2!}\sum_0^\infty (n+1)z^{n+2}+\frac{1}{3!}\sum_0^\infty\frac{(n+2)(n+1)}{2!}z^{n+3}+\ldots\right.$$
$$\left.\ldots+\frac{1}{k!}\sum_0^\infty\binom{k+n-1}{k-1}z^{n+k}+\ldots\right] =$$
$$= e\left\{1+z+\left(\frac{1}{1!}+\frac{1}{2!}\right)z^2+\left(\frac{1}{1!}+2\frac{1}{2!}+\frac{1}{3!}\right)z^3+\left(\frac{1}{1!}+3\frac{1}{2!}+3\frac{1}{3!}+\frac{1}{4!}\right)z^4+\ldots\right.$$
$$\left.\ldots+\left[\frac{1}{1!}+\binom{n-1}{1}\frac{1}{2!}+\binom{n-1}{2}\frac{1}{3!}+\ldots+\binom{n-1}{n-2}\frac{1}{(n-1)!}+\frac{1}{n!}\right]z^n+\ldots\right\}.$$

This is the desired expansion. The function $f(z)$ is analytic in the circle $|z| < 1$. Hence, the series obtained converges in the unit circle. Since all the coefficients of the series within the braces are positive numbers not less than unity, the series diverges for $z = 1$. Hence, it follows that the radius of convergence of the series is unity.

10. DIVISION OF THE POWER SERIES

We now pass on to the question of the *division of the power series.* Let

$$a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots \tag{22}$$

and

$$b_0 + b_1(z-a) + \ldots + b_n(z-a)^n + \ldots \tag{23}$$

be two power series with positive radii of convergence r and ρ, the free term b_0 in the second series being different from zero. We denote by σ the smaller of the two numbers r and ρ : $\sigma = \min(r, \rho)$ (if $r=\rho$, then $\sigma=r=\rho$). Then, both the series converge in the circle $|z-a| < \sigma$. If any zeros of the sum of the series (23) are contained in this circle, we take a new circle of smaller radius in the interior of which the sum of this series does not vanish (such a circle actually exists because the point a is not a zero for the sum of the series (23), by virtue of the condition $b_0 \neq 0$). Thus, there exists a circle $|z-a| < R$ in which both the series converge and the sum of the second series has no zero. In the interior of this circle the ratio

$$f(z) = \frac{a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots}{b_0 + b_1(z-a) + \ldots + b_n(z-a)^n + \ldots} \tag{24}$$

represents an analytic function ; this follows from the rule of differentiation of a quotient. Hence, there exists a power series

$$c_0 + c_1(z-a) + \ldots + c_n(z-a)^n + \ldots, \tag{25}$$

representing the function $f(z)$ in the interior of the circle $|z-a| < R$. We refer to this series as the *quotient* of the series (22) (*dividend*) and (23) (*divisor*), and the process of determining this quotient the *division of series.*

We now perform the division of series, first by the method of *undetermined coefficients.* For this purpose we rewrite the ratio (24) in the form

$$[c_0 + c_1(z-a) + \ldots + c_n(z-a)^n + \ldots] \cdot [b_0 + b_1(z-a) + \ldots + b_n(z-a)^n + \ldots] = a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots$$

and note that the power series in question, convergent in the interior of

the circle $|z-a|<R$, must converge in it absolutely. Hence, the series on the left-hand side of the last equation must be multiplied out term by term. Thus, one obtains

$$\begin{aligned} c_0b_0+(c_0b_1+c_1b_0)(z-a)+(c_0b_2+c_1b_1+c_2b_0)(z-a)^2+\ldots \\ \ldots+(c_0b_n+c_1b_{n-1}+\ldots+c_nb_0)(z-a)^n+\ldots= \\ =a_0+a_1(z-a)+\ldots+a_n(z-a)^n+\ldots\ . \end{aligned} \tag{26}$$

As the sums of the power series on the left- and right-hand sides here coincide in the circle $|z-a|<R$, it follows, by the uniqueness property of the power series, that the coefficients of the two series the must be same.

Thus we obtain the equation

$$\left.\begin{aligned} &c_0b_0=a_0, \\ &c_0b_1+c_1b_0=a_1, \\ &c_0b_2+c_1b_1+c_2b_0=a_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &c_0b_n+c_1b_{n-1}+c_2b_{n-2}+\ldots+c_nb_0=a_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{27}$$

This is an infinite system of linear equations for the unknown coefficients $c_0, c_1, c_2, \ldots c_n, \ldots$. The peculiarity of this system, which actually facilitates its solution, lies in that, for any n ($n=0, 1, 2, 3, \ldots$), the first $n+1$ equations contain just the first $(n+1)$ unknowns. From the first equation we find $c_0=\frac{a_0}{b_0}$ ($b_0\neq 0$, by assumption). Substituting this in the second equation we obtain

$$\frac{a_0}{b_0}b_1+c_1b_0=a_1,$$

whence

$$c_1=\frac{a_1b_0-a_0b_1}{b_0^2}.$$

Suppose, in general, that we have already determined the first n coefficients $c_0, c_1, \ldots, c_{n-1}$. Substituting them in the $(n+1)$-th equation, we obtain

$$c_n=\frac{a_n-c_0b_n-c_1b_{n-1}-\ldots-c_{n-1}b_1}{b_0}. \tag{28}$$

Thus, we can determine coefficients with any preassigned subscript. It is easy to obtain the expression for c_n in terms of the coefficients $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ in the form of a determinant. The determinant

of the system formed by the first $n+1$ equations is

$$\begin{vmatrix} b_0 & 0 & 0 & \dots & 0 \\ b_1 & b_0 & 0 & \dots & 0 \\ b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_n & b_{n-1} & b_{n-2} & \dots & b_0 \end{vmatrix} = b_0^{n+1} \neq 0,$$

hence

$$c_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & 0 & 0 & a_0 \\ b_1 & b_0 & 0 & a_1 \\ b_2 & b_1 & b_0 & a_2 \\ \dots & \dots & \dots & \dots \\ b_n & b_{n-1} & b_{n-2} & \dots a_n \end{vmatrix}. \tag{29}$$

This formula solves completely the problem of the division of series.

We shall now show that the quotient (25) of the two power series can be obtained by dividing the series (22) by the series (23) in a manner as if the series (22) and (23) were polynominals with ascending powers of $(z-a)$. To prove this, let us perform such a division. We thereby obtain

$$\begin{array}{l|l} a_0+a_1(z-a)+\dots+a_n(z-a)^n+\dots & b_0+b_1(z-a)\dots+b_n(z-a)^n+\dots \\ \underline{a_0+\frac{a_0}{b_0}b_1(z-a)+\dots+\frac{a_0}{b_0}b_n(z-a)^n+\dots} & \frac{a_0}{b_0}+\frac{a_1b_0-a_0b_1}{b_0^2}(z-a)+\dots \\ \frac{a_1b_0-a_0b_1}{b_0}(z-a)+\dots+\frac{a_nb_0-a_0b_n}{b_0}(z-a)^n+\dots & \\ \underline{\frac{a_1b_0-a_0b_1}{b_0}(z-a)+\dots+\frac{a_1b_0-a_0b_1}{b_0^2}b_{n-1}(z-a)^n+\dots} & \end{array}$$

The first two coefficients of the quotient obtained here agree with the values of c_0 and c_1 obtained above from equations (27). We assume to have obtained in this way the first n coefficients of the quotient, agreeing with the values of $c_0, c_1, \dots c_{n-1}$ obtained from equations (27). Then, we would have

$$\begin{array}{l|l} a_0+a_1(z-a)+a_2(z-a)^2+\dots+a_n(z-a)^n+\dots & b_0+b_1(z-a)+\dots+b_n(z-a)^n+\dots \\ \underline{b_0c_0+b_1c_0(z-a)+b_2c_0(z-a)^2+\dots+b_nc_0(z-a)^n+\dots} & c_0+c_1(z-a)+\dots+c_{n-1}(z-a)^n+\dots \\ (a_1-b_1c_0)(z-a)+ & \\ +(a_2-b_2c_0)(z-a)^2+\dots & \\ \dots+(a_n-b_nc_0)(z-a)^n+\dots & \text{1st remainder} \\ b_0c_1(z-a)+b_1c_1(z-a)^2+\dots & \\ \underline{\dots+b_{n-1}c_1(z-a)^n+\dots} & \\ (a_2-b_2c_0-b_1c_1)(z-a)^2+\dots & \\ \dots+(a_n-b_nc_0-b_{n-1}c_1)(z-a)^n+\dots & \text{2nd remainder} \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ \underline{(a_n-b_nc_0-b_{n-1}c_1-\dots-b_1c_{n-1})(z-a)^n+\dots} & n\text{-th remainder} \\ \dots\dots\dots\dots\dots\dots\dots\dots & \end{array}$$

The first term of the n-th remainder is $(a_n - b_n c_0 - b_{n-1} c_1 - \ldots - b_1 c_{n-1})(z-a)^n$; hence, the term following $c_{n-1}(z-a)^{n-1}$ in the quotient is

$$\frac{a_n - b_n c_0 - b_{n-1} c_1 - \ldots - b_1 c_{n-1}}{b_0} (z-a)^n.$$

But the coefficient in this term agrees with the value of c_n obtained by formula (28) from the equations (27).

Thus, *the method of undetermined coefficients, when applied to division of the power series, leads to the same result as does the operation performed according to the rule of division of the polynomials arranged in ascending powers of* $x = z - a$.

We give here an example of the division of power series. Let us take the function

$$F(z) = \frac{z}{e^z - 1}.$$

This function is analytic at all the points of the plane excluding the zeros of the function $e^z - 1$, i.e. excluding the points $0, \pm 2\pi i, \pm 4\pi i, \ldots$ Replacing $e^z - 1$ by its expansion

$$e^z - 1 = \frac{z}{1} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

and dividing both the numerator and the denominator of the fraction by z, we obtain the following expression for $F(z)$ (which defines the function $F(z)$ for $z = 0$ also):

$$F(z) = \frac{1}{1 + \frac{z}{2!} + \ldots + \frac{z^n}{(n+1)!} + \ldots}.$$

The series in the denominator converges for any z and has the same zeros as the function $e^z - 1$, excluding the one at the origin of coordinates. [All this follows from the fact that this series represents the function $\frac{e^z - 1}{z}$ (for $z \neq 0$)]. Hence its sum does not vanish in the interior of the circle $|z| < 2\pi$. Consequently, the function $F(z)$ can be expanded into a series in this circle by means of a division of series. The first of the equations (27) gives

$$c_0 \cdot 1 = 1, \quad \text{i.e.} \quad c_0 = 1.$$

Since all the coefficients of the series for the dividend (except the

initial one) are zero, the $(n+1)$-th equation (27) has the form:

$$c_0 \frac{1}{(n+1)!} + c_1 \frac{1}{n!} + \ldots + c_{n-1} \frac{1}{2!} + c_n = 0 \quad (n=1, 2, 3, \ldots). \tag{30}$$

This equation enables us to determine the numbers c_n successively. For determining the coefficients c_n, we can also make use of the formula (29):

$$c_0 = 1, \; c_n = \begin{vmatrix} 1 & 0 & 0 & \ldots 1 \\ \frac{1}{2!} & 1 & 0 & \ldots 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(n-1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \ldots 0 \end{vmatrix} =$$

$$= (-1)^n \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \ldots 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \ldots \frac{1}{2!} \end{vmatrix} \quad (n=1, 2, 3, \ldots).$$

The numbers $c_n n!$ are called the *Bernoulli numbers* and are denoted by B_n: $B_n = c_n n!$.

The coefficients of several important series are easily expressed in terms of these numbers. For their calculation we have the formulae: $B_0 = c_n \cdot 0! = 1$,

$$B_n = c_n n! = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \ldots 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \ldots \frac{1}{2!} \end{vmatrix} \quad (n=1, 2, 3 \ldots). \tag{31}$$

The calculation of the successive Bernoulli numbers can, however, be easily done with the help of the formula (30). We obtain

$$B_0 \cdot \frac{1}{0!\,(n+1)!} + B_1 \frac{1}{1!\,n!} + \ldots + B_n \frac{1}{n!\,1!} = 0 \quad (n=1, 2, 3, \ldots),$$

or, multiplying both sides of the equality by $(n+1)!$ and observing that $\frac{(n+1)!}{k!\,(n+1-k)!}$ is just the binomial coefficient $\binom{n+1}{k}$,

$$B_0 \binom{n+1}{0} + B_1 \binom{n+1}{1} + \ldots + B_n \binom{n+1}{n} = 0 \quad (n=1, 2, 3, \ldots).$$

This formula can be written in the following symbolic form :

$$(1+B)^{n+1} - B^{n+1} = 0. \tag{32}$$

After involution by means of the binomial formula, all the exponents here are to be replaced by subscripts.

Since $B_0 = 1$, we obtain in turn

$$B_0 + 2B_1 = 0\,;\ B_1 = -\frac{1}{2}B_0 = -\frac{1}{2}\,;$$

$$B_0 + 3B_1 + 3B_2 = 0\,;\ B_2 = -\frac{1}{3}B_0 - B_1 = \frac{1}{6}\,;$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0\,;\ B_3 = -\frac{1}{4}B_0 - B_1 - \frac{3}{2}B_2 = 0\,;$$

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0\,;$$

$$B_4 = -\frac{1}{5}B_0 - B_1 - 2B_2 - 2B_3 = -\frac{1}{30}\,;$$

$$B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5 = 0\,;$$

$$B_5 = -\frac{1}{6}B_0 - B_1 - \frac{5}{2}B_2 - \frac{10}{3}B_3 - \frac{5}{2}B_4 = 0\,;$$

$$B_0 + 7B_1 + 21B_2 + 35B_3 + 35B_4 + 21B_5 + 7B_6 = 0\,;$$

$$B_6 = -\frac{1}{7}B_0 - B_1 - 3B_2 - 5B_3 - 5B_4 - 3B_5 = \frac{1}{42}\,;$$

. . . . . .

Thus,

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42} \ldots .$$

Now we shall show that *all Bernoulli numbers with odd subscripts, greater than unity, are equal to zero* :

$$B_{2k+1} = 0 \quad (k=1, 2, 3 \ldots).$$

To prove this, we replace z by $-z$ in the expansion

$$\frac{z}{e^z-1} = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots =$$

$$= B_0 + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \ldots + \frac{B_n}{n!} z^n + \ldots, \quad (33)$$

and obtain

$$\frac{-z}{e^{-z}-1} = -\frac{ze^z}{(e^{-z}-1)e^z} = \frac{ze^z}{e^z-1} = B_0 - \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 - \frac{B_3}{3!} z^3 + \ldots .$$

Subtracting the last equation from (33), we obtain

$$\frac{z}{e^z-1} - \frac{ze^z}{e^z-1} = -z = 2\frac{B_1}{1!} z + 2\frac{B_3}{3!} z^3 + \ldots + 2\frac{B_{2k+1}}{(2k+1)!} z^{2k+1} + \ldots$$

Hence, by the principle of uniqueness of expansion into power series it follows that

$$2B_1 = -1, \quad B_3 = B_5 = \ldots = B_{2k+1} = \ldots = 0,$$

as was to be proved.

Using the above-mentioned property of the Bernoulli numbers, we can rewrite the expansion (33) in the form :

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}. \quad (34)$$

18. EXPANSION OF THE FUNCTIONS cot z, tan z, cosec z and sec z INTO POWER SERIES

From the expansion (34) we can readily derive expansions for the functions $z \cot z$, $\tan z$ and $z \operatorname{cosec} z$. We write $\cot z$ in the form

$$\cot z = \frac{\cos z}{\sin z} = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{e^{2iz}+1}{e^{2iz}-1} = i + \frac{2i}{e^{2iz}-1},$$

whence

$$z \cot z = iz + \frac{2iz}{e^{2iz}-1}.$$

The function $\frac{2iz}{e^{2iz}-1}$ can be expanded with the help of the formula (34) if we replace z there by $2iz$. Since the series (34) converges for $|z| < 2\pi$,

hence, the obtained series will converge for $|2iz| < 2\pi$, i.e. for $|z| < \pi$. Thus

$$\frac{2iz}{e^{2iz}-1} = 1 - \frac{2iz}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (2iz)^{2k} = 1 - iz + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}B_{2k}}{(2k)!} z^{2k}$$

and, consequently,

$$z \cot z = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}B_{2k}z^{2k}}{(2k)!}. \tag{35}$$

To obtain the Taylor expansion of $\tan z$, we simply note that

$$\cot z - \tan z = 2 \cot 2z,$$

whence

$$\tan z = \cot z - 2 \cot 2z.$$

Replacing z by $2z$ in the formula (35), we obtain a series which converges for $|2z| < \pi$, i.e. for $|z| < \frac{\pi}{2}$:

$$2z \cot 2z = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{4k}B_{2k}}{(2k)!} z^{2k}. \tag{36}$$

Subtracting (36) from (35) termwise, we get

$$z \cot z - 2z \cot 2z = z \tan z = \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}(1-2^{2k})B_{2k}}{(2k)!} z^{2k}$$

or

$$\tan z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} z^{2k-1}. \tag{37}$$

From the mode of derivation of this series, it follows that it converges for $|z| < \frac{\pi}{2}$.

Passing on to the function z cosec z, we observe that

$$\cot z + \tan \frac{z}{2} = \frac{\cos z \cos \frac{z}{2} + \sin z \sin \frac{z}{2}}{\sin z \cos \frac{z}{2}} = \frac{\cos \frac{z}{2}}{\sin z \cos \frac{z}{2}} = \operatorname{cosec} z.$$

Replacing z by $\frac{z}{2}$ in (37), we obtain a series that converges for $\left|\frac{z}{2}\right| < \frac{\pi}{2}$, i.e. for $|z| < \pi$. Hence, from expansion (35), which also converges for $|z| < \pi$, we derive :

$$z \operatorname{cosec} z = z \cot z + z \tan \frac{z}{2} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2^{2k}-2)\, B_{2k}}{(2k)!} z^{2k}. \quad (38)$$

Finally, we obtain an expansion for $\sec z$. Since this function is analytic in the circle $|z| < \frac{\pi}{2}$, the required expansion will converge in the same circle. Using again the method of division of series, we have

$$\sec z = \frac{1}{\cos z} = \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + (-1)^k \frac{z^{2k}}{(2k)!} + \ldots} =$$

$$= a_0 + a_1 z + a_2 z^2 + \ldots .$$

Since $\sec z$ is an even function, all the coefficients of odd powers, $a_1, a_3, a_5, \ldots$, must vanish and, hence,

$$\sec z = \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots} = a_0 + a_2 z^2 + a_4 z^4 + \ldots . \quad (39)$$

It is customary to write the coefficients of this expansion in the form

$$a_{2k} = (-1)^k \frac{E_{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots).$$

The numbers E_{2k}, as determined by this method, are referred to as the *Euler numbers*. Rewriting (39) in the form

$$1 = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right)\left(E_0 - \frac{E_2}{2!} z^2 + \frac{E_4}{4!} z^4 - \ldots\right)$$

and multiplying out the right-hand side, we obtain

$$1 = E_0 - \left(\frac{E_0}{2!} + \frac{E_2}{2!}\right) z^2 +$$

$$+ \left(\frac{E_0}{4!} + \frac{E_2}{2!\,2!} + \frac{E_4}{4!}\right) z^4 - \left(\frac{E_0}{6!} + \frac{E_2}{4!\,2!} + \frac{E_4}{2!\,4!} + \frac{E_6}{6!}\right) z^6 + \ldots,$$

whence

$$E_0 = 1,$$
$$\frac{E_0}{2!} + \frac{E_2}{2!} = 0,$$
$$\frac{E_0}{4!} + \frac{E_2}{2!\,2!} + \frac{E_4}{4!} = 0,$$
$$\frac{E_0}{6!} + \frac{E_2}{4!2!} + \frac{E_4}{2!4!} + \frac{E_6}{6!} = 0,$$
$$\cdots\cdots\cdots\cdots$$

These equations enable us to determine in succession the Euler numbers; we obtain

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385, \ldots .$$

In general, if the numbers $E_0, E_2, \ldots, E_{2n-2}$ have already been determined, then for determining E_{2n} we have the equality

$$\frac{E_0}{(2n)!} + \frac{E_2}{(2n-2)!2!} + \frac{E_4}{(2n-4)!4!} + \ldots + \frac{E_{2n}}{(2n)!} = 0,$$

or

$$E_0 + \binom{2n}{2} E_2 + \binom{2n}{4} E_4 + \ldots + \binom{2n}{2n-2} E_{2n-2} + E_{2n} = 0.$$

It thus follows that if the numbers $E_0, E_2, \ldots, E_{2n-2}$ are integers, then the number E_{2n} will also be an integer. But the initial numbers obtained above are integers; hence, all the Euler numbers are integers.

We finally write down the expansion of sec z in the form

$$\sec z = \sum_0^\infty (-1)^k \frac{E_{2k}}{(2k)!} z^{2k}. \tag{40}$$

This converges for $|z| < \frac{\pi}{2}$. All the Euler numbers E_{2k}, appearing in the coefficients of this expansion, can be completely determined from the relations

$$E_0 = 1,\ E_0 + \binom{2n}{2} E_2 + \binom{2n}{4} E_4 + \ldots + \binom{2n}{2n-2} E_{2n-2} + E_{2n} = 0 \tag{41}$$
$$(n=1, 2, 3, \ldots).$$

12. EXPANSION OF HARMONIC FUNCTIONS INTO SERIES. THE POISSON INTEGRAL AND THE SCHWARTZ FORMULA

Using the relation between harmonic functions and analytic functions, we can derive the fundamental properties of the harmonic functions from those already known for the analytic functions.

Let $u(x, y)$ be a one-valued function, harmonic in some circle $K: |z-z_0| < R$. Then, by Art. 12 of Chapter 2, there exists in the circle K a one-valued harmonic function $v(x, y)$ conjugate to $u(x, y)$. We form a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$ and expand it as a power series in $z - z_0$. Thus, we obtain

$$f(z) = \sum_0^\infty (\alpha_n + i\beta_n)(z-z_0)^n, \tag{42}$$

where α_n and β_n are real numbers. We now introduce polar coordinates r and θ, with pole at the point z_0, so that $z = z_0 + re^{i\theta}$, and use for $u(x, y)$ and $v(x, y)$ the notation $u(r, \theta)$ and $v(r, \theta)$ as equivalent to $u(x, y)$ and $v(x, y)$. Separating out the real and imaginary parts of (42), we obtain the series

$$u(r, \theta) = \alpha_0 + \sum_1^\infty (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n, \tag{43}$$

$$v(r, \theta) = \beta_0 + \sum_1^\infty (\beta_n \cos n\theta + \alpha_n \sin n\theta) r^n, \tag{44}$$

which converge uniformly in the interior of K.

Thus, *every function* $u(r, \theta)$, *harmonic in the interior of the circle* $|z - z_0| < R$, *admits an expansion into the form* (43) *which converges uniformly in the interior of this circle.* The coefficients of the series are the real numbers α_n and $-\beta_n$. Since the power series (42) converges in the given circle and might have a radius of convergence greater than R, these numbers must satisfy the inequality

$$R \leqslant \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|\alpha_n + i\beta_n|}}.$$

It is easy to see that if α_n and β_n are given *a'priori* and satisfy the last inequality, the series (43) will define a function harmonic in the interior of the circle K. To be convinced of this, it is sufficient to observe that when the above-mentioned inequality is satisfied, the power series (42)

converges in the circle K and, hence, represents a function which is analytic in it and whose real part is also expressed by the given series (43). Thus, *the existence of an expansion in the form* (43) *or* (44), *with corresponding inequality imposed on the coefficients of the series, is a characteristic of the functions which are harmonic in the interior of the given circle.*

We apply the above considerations to the functions $f(z)=\frac{\rho e^{i\alpha}+(z-z_0)}{\rho e^{i\alpha}-(z-z_0)}$ whose expansion in the circle $|z-z_0|<\rho$ has the form :

$$\frac{\rho e^{i\alpha}+(z-z_0)}{\rho e^{i\alpha}-(z-z_0)} = -1+\frac{2\rho e^{i\alpha}}{\rho e^{i\alpha}-(z-z_0)} =$$

$$= -1+2\left[1+\frac{z-z_0}{\rho e^{i\alpha}}+\frac{(z-z_0)^2}{\rho^2 e^{2i\alpha}}+\dots\right] = 1+2\sum_1^\infty \frac{(z-z_0)^n}{\rho^n}e^{-in\alpha}.$$

We obtain[1]

$$\operatorname{Re} f(z) = \frac{\rho^2-r^2}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)} = 1+2\sum_1^\infty \left(\frac{r}{\rho}\right)^n \cos n(\theta-\alpha), \quad (45)$$

$$\operatorname{Im} f(z) = \frac{2r\rho\sin(\theta-\alpha)}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)} = 2\sum_1^\infty \left(\frac{r}{\rho}\right)^n \sin n(\theta-\alpha); \quad (46)$$

here, both the series converge uniformly in the interior of the circle $|z-z_0|<\rho$.

We now turn to the series (43) and (44). In the first of these we replace r by an arbitrary member ρ ($\rho<R$), θ by α and thereafter multiply both its sides by $\cos m\alpha$ and integrate (with ρ fixed) over α, between the limits zero and 2π.

Thus, we obtain

$$\int_0^{2\pi} u(\rho,\alpha)\cos m\alpha\, d\alpha = \alpha_m \rho^m \int_0^{2\pi}\cos^2 m\alpha\, d\alpha,$$

whence

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\alpha)\,d\alpha,\quad \alpha_m = \frac{1}{\pi\rho^m}\int_0^{2\pi} u(\rho,\alpha)\cos m\alpha\, d\alpha \quad (m\geqslant 1). \quad (47)$$

[1] We obtain the real and imaginary parts of $f(z)$ by multiplying the numerator and the denominator by an expression conjugate to the denominator :

$$\frac{\rho e^{i\alpha}+re^{i\theta}}{\rho e^{i\alpha}-re^{i\theta}} = \frac{(\rho e^{i\alpha}+re^{i\theta})(\rho e^{-i\alpha}-re^{-i\theta})}{(\rho e^{i\alpha}-re^{i\theta})(\rho e^{-i\alpha}-re^{-i\theta})} = \frac{\rho^2-r^2+i2r\rho\sin(\theta-\alpha)}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)}.$$

Similarly, multiplying by $\sin m\alpha$ and integrating between the same limits, we obtain

$$-\beta_m = \frac{1}{\pi\rho^m}\int_0^{2\pi} u(\rho, \alpha)\sin m\alpha\, d\alpha \quad (m \geqslant 1). \tag{48}$$

Substituting the obtained expression for α_n and β_n in the series (43) and (44), we have

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\, d\alpha + \sum_1^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho, \alpha)\cos m(\theta-\alpha)\, d\alpha \left(\frac{r}{\rho}\right)^m,$$

and

$$v(r, \theta) = \beta_0 + \sum_1^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho, \alpha)\sin m(\theta-\alpha)\, d\alpha \left(\frac{r}{\rho}\right)^m.$$

Let ρ satisfy the condition $r < \rho < R$. Evidently, the foregoing formluae can be derived from (45) and (46) by multiplying them with $\frac{1}{2\pi}u(\rho, \alpha)$ and then integrating termwise over α between the limits 0 and 2π (with r and ρ fixed). All these operations are justified because of the uniform convergence of the series (45) and (46) in the interior of the circle $|z-z_0| < \rho$.

Thus, we obtain

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\left[1 + 2\sum_1^{\infty}\left(\frac{r}{\rho}\right)^n \cos n(\theta-\alpha)\right] d\alpha =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\alpha)}\, d\alpha, \tag{49}$$

and

$$v(r, \theta) = \beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\left[2\sum_1^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta-\alpha)\right] d\alpha =$$

$$= \beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\frac{2r\rho\sin(\theta-\alpha)}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)}\, d\alpha. \tag{50}$$

We have derived for the function $u(r, \theta)$ and its conjugate function $v(r, \theta)$ integral expressions in terms of the values of $u(\rho, \alpha)$ at the points of the circle $|z-z_0| = \rho_0 < R$.

The difference in the form of the integrals in the formulae (49) and (50) is explained by the fact that in the second of these the harmonic function $v(r, \theta)$ is expressed not through its characteristic values on the circle $|z-z_0| = \rho$ but through the values of the function conjugate to it. However, the formula (49), established for any function harmonic in the given circle, is also true for $v(x, y)$. Hence, for $v(r, \theta)$ we have a formula similar to that for $u(r, \theta)$:

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} v(\rho, \alpha) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha. \tag{51}$$

The integral on the right-hand side of (49) or (51) is called the *Poisson integral* of the corresponding function $u(\rho, \alpha)$ or of $v(\rho, \alpha)$, and the harmonic function

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} = \operatorname{Re}\left[\frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)}\right]$$

is called the *kernel of this integral.*

Assuming $u(r, \theta) \equiv 1$ in the formula (49), we obtain

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha. \tag{52}$$

In general, if $\varphi(\alpha)$ is a real function, defined and continuous on the segment $[0, 2\pi]$, then we shall call an expression of the form

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(\alpha) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha \tag{53}$$

a Poisson integral, without requiring the function $\varphi(\alpha)$ to coincide in value with some harmonic function $u(\rho, \alpha)$. We can prove that the integral (53) represents a harmonic function in the circle $|z-z_0| < \rho$, tending to the limit $\varphi(\alpha)$[1] when the point (r, θ) approaches the point (ρ, α).

[1] See, in this connection, A. I. Markushevich, *Theory of Analytic Functions*, Gostekhizdat, 1950, Chap. VI, Art. 1.5.

The Poisson integral is analogous to the Cauchy integral extended to the circle and can be obtained by certain transformations of the latter. With this object we examine, along with the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{54}$$

where the point z lies in the interior of the circle $|\zeta - z_0| = \rho_0$, the Cauchy integral, as obtained by replacing z by the point $z^* = z_0 + \frac{\rho^2}{z - z_0}$, symmetric with z relative to the circle $|\zeta - z_0| = \rho$. Since the point z^* lies in the exterior of the circle, this integral must be equal to zero :

$$0 = \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{\zeta - z^*}\, d\zeta. \tag{55}$$

We now subtract (55) from (54) term by term and transform the result, using the relations

$$\zeta - z = \zeta - z_0 - (z - z_0) = \rho e^{i\alpha} - re^{i\theta},$$

$$\zeta - z^* = \zeta - z_0 - (z^* - z_0) = \rho e^{i\alpha} - \frac{\rho^2}{r} e^{i\theta} \text{ and } d\zeta = i\rho e^{i\alpha}\, d\alpha.$$

Thus, we obtain

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = \rho} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left[\frac{\rho}{\rho - re^{i(\theta - \alpha)}} + \frac{re^{i(\alpha - \theta)}}{\rho - re^{i(\alpha - \theta)}} \right] d\alpha =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho \cos(\alpha - \theta)}\, d\alpha.$$

Replacing $f(z)$ by $u(r, \theta) + iv(r, \theta)$ and $f(\zeta)$ by $u(\rho, \alpha) + iv(\rho, \alpha)$ and separating out the real and imaginary parts, we again obtain the formulae (49) and (51).

From the formulae (49) and (50) we can readily derive an important formula which expresses the analytic function $f(z)$ in terms of the values of its real part on a circle. In fact, multiplying (50) by i and adding the

result to (49), we obtain

$$f(z) = i\beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\left[\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho\cos(\theta-\alpha)} + \right.$$

$$\left. + i\,\frac{2r\rho\sin(\theta-\alpha)}{\rho^2 + r^2 - 2r\rho\cos(\theta-\alpha)}\right] d\alpha.$$

But the expression within the square brackets represents an analytic function of z:

$$\frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)}.$$

Hence

$$f(z) = i\beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\,\frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)}\,d\alpha. \tag{56}$$

Here, $i\beta_0$ is a purely imaginary constant representing the imaginary part of $f(z_0)$; this constant, however, cannot be determined by the real part of the function $f(z)$. Formula (56) is called the *Schwartz formula.*

From (49) we obtain, for $r = 0$, i.e. for the centre of the circle K:

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\,d\alpha,$$

where $u(\rho, \alpha)$ stands, as assumed, for the values of the function $u(x, y)$ at the points of the circle $|z - z_0| = \rho$ with its centre at $z_0 = x_0 + iy_0$. If we write out in detail, we shall get

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + \rho\cos\alpha,\ y_0 + \rho\sin\alpha)\,d\alpha. \tag{57}$$

Thus, *the value of a harmonic function at the centre of a circle is equal to the arithmetic mean of its values on the periphery of the circle.* We can show that this last property is a characteristic property of harmonic functions. More precisely, the following proposition is valid.

Let $u(x, y)$ be a real function, one-valued and continuous in the domain G. If, for every point $z_0 = x_0 + iy_0 \in G$, there exists a neighbourhood $| z - z_0 | < \delta(z_0)$ in which $u(x_0, y_0)$ is equal to the arithmetic mean of its values on any circle $| z - z_0 | = \rho$ $[0 < \rho < \delta(z_0)]$, then $u(x, y)$ is a harmonic function in the domain G.[1]

[1] See, in this connection, A.I. Markushevich, loc. cit., Gostekhizdat, 1950, Chap. VI, Art. 1.3.

7

THE LAURENT SERIES. ISOLATED SINGULAR POINTS OF A SIMPLE CHARACTER. ENTIRE AND MEROMORPHIC FUNCTIONS

1. THE LAURENT SERIES

Among the classes of series of analytic functions there are series, although quite distinct from, but very similar to, the power series, in their origin and properties, which can be decomposed into negative integral powers of $z - z_0$:

$$A_0 + A_1 (z-z_0)^{-1} + A_2 (z-z_0)^{-2} + \ldots + A_n (z-z_0)^{-n} + \ldots \quad . \quad (1)$$

Putting $\zeta = \dfrac{1}{z-z_0}$, we transform this series into

$$A_0 + A_1\zeta + A_2\zeta^2 + \ldots + A_n\zeta^n + \ldots \quad . \quad (2)$$

The radius of convergence of the latter series is $R = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}}$; if $R = 0$, the series (2) converges only at the point $\zeta = 0$; if $0 < R < \infty$, the series converges absolutely in the circle $|\zeta| < R$ and diverges outside it. Finally, if $R = \infty$, the series converges absolutely at every finite point of the plane. Hence, by virtue of the relation $|\zeta| = \dfrac{1}{|z-z_0|}$, it follows that if $\overline{\lim} \sqrt[n]{|A_n|} = \infty$, the series (1) diverges at every finite point; if $0 < \overline{\lim} \sqrt[n]{|A_n|} < \infty$, it converges absolutely for $|z-z_0| > \overline{\lim} \sqrt[n]{|A_m|}$ and diverges for $|z-z_0| < \overline{\lim} \sqrt[n]{|A_n|}$; finally, if $\sqrt[n]{|A_n|} = 0$, the series converges absolutely at all the points of the plane, excluding the point $z = z_0$. In other words, the domain of convergence of the series (1) is the *exterior of a circle* of radius $r = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}$ with its centre at z_0. This domain degenerates to the point at infinity for $r = \infty$, it is the exterior of the circle in the true sense of the term

for $0 < r < \infty$ and, finally, it turns into the whole plane (excluding the point $z = z_0$) for $r = 0$. We shall take $r = \varlimsup \sqrt[n]{|A_n|} < \infty$. Then, there actually exists a domain of convergence of the series (1): $|z - z_0| > r$, and we shall denote this domain by K. Since the series (2) converges uniformly in every closed point set of the circle k: $|\zeta| < R$, and the linear transformation $\zeta = \dfrac{1}{z - z_0}$ transforms any closed point set of the circle k into a certain closed point set of the domain K and conversely; hence, the series (1) converges uniformly in the interior of the domain K. It defines the function $F(z)$:

$$F(z) = A_0 + A_1 (z-z_0)^{-1} + \ldots + A_n (z-z_0)^{-n} + \ldots \qquad (1')$$

in this domain; this function is analytic (by the theorem of Weierstrass) at all finite points of the domain K. It takes up the value A_0: $F(\infty) = A_0$, at the point at infinity. We shall, by definition, call the function $F(z)$ *analytic at the point at infinity*. Thus, the analyticity of the function at the point at infinity will be characterized by the existence of an expansion of the form (1'), converging in some neighbourhood of the point at infinity.

The *Laurent series* is a generalisation of the concept of a series expanded only in nonnegative integral powers of $z - z_0$ (a power series) or one expanded only in nonpositive integral powers of $z - z_0$, namely the series

$$\sum_{-\infty}^{+\infty} a_n (z-z_0)^n. \qquad (3)$$

This series is the sum of the two series

$$\sum_{0}^{\infty} a_n (z-z_0)^n \text{ and } \sum_{1}^{\infty} a_{-m} (z-z_0)^{-m} \qquad (4)$$

and can be regarded as convergent if, and only if, both the series (4) converge. Thus, by definition,

$$\sum_{-\infty}^{+\infty} a_n (z-z_0)^n = \lim_{\nu \to \infty} \sum_{0}^{\nu} a_n (z-z_0)^n + \lim_{\mu \to \infty} \sum_{1}^{\mu} a_{-m} (z-z_0)^{-m},$$

or, what is the same thing,

$$\sum_{-\infty}^{+\infty} a_n (z-z_0)^n = \lim_{\substack{\nu \to \infty \\ \mu \to \infty}} \sum_{-\mu}^{\nu} a_n (z-z_0)^n. \qquad (5)$$

Here, μ and ν tend to infinity independently of each other. The last expression implies that for any $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that the inequality

$$\left| \sum_{-\infty}^{+\infty} a_n (z-z_0)^n - \sum_{-\mu}^{\nu} a_n (z-z_0)^n \right| < \varepsilon$$

is satisfied for $\nu > N(\varepsilon)$ and $\mu > N(\varepsilon)$.

The properties of absolute and uniform convergence of a Laurent series are, by definition, identical to those of the series (4).

We denote $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$ by λ and $\overline{\lim_{m\to\infty}} \sqrt[m]{|a_{-m}|}$ by r. Then the first of the series (4) converges absolutely and uniformly in the interior of the domain G representing the interior of the circle $\Gamma : |z - z_0| = \frac{1}{\lambda} = R$, and diverges in its exterior, while the second of the series (4) converges absolutely and uniformly in the interior of the domain g representing the exterior of the circle $\gamma : |z-z_0| = r$ and diverges in its interior.

The domains G and g have common points, if, and only if, the inequality

$$r < R \tag{6}$$

is satisfied. In this case, the part common to the domains G and g would be the *circular ring* D :

$$r < |z-z_0| < R. \tag{7}$$

Both the series converge absolutely and uniformly in the interior of the domain (7); hence, the Laurent series (3), representing in D the analytic function

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z-z_0)^n \quad (r<|z-z_0|<R), \tag{8}$$

also converges absolutely and uniformly in the interior of this domain.

At every point outside this domain one of the series (4) diverges while the other continues to converge. Hence, it follows that the Laurent series diverges outside the domain D. Thus, *the domain of convergence of a Laurent series is a circular ring*[1] under the condition (6). Later, while speaking of the Laurent series, we shall always assume that

[1] Besides this, the Laurent series can also converge at some point lying on the boundary of the circular ring.

the condition (6) is fulfilled except for the case when no domain of convergence of the series can exist.

If $r < \rho < R$, the series (8) converges uniformly on the circle γ : $|z - z_0| = \rho$. It will also converge uniformly on γ when all its terms are multiplied by $\frac{1}{2\pi i}(z-z_0)^{-k-1}$, k being an arbitrary integer. Integrating the resulting series along γ termwise, we get

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{-\infty}^{+\infty} a_n \frac{1}{2\pi i}\int_{\gamma}(z-z_0)^{n-k-1}\,dz.$$

All the integrals on the right-hand side, as a simple calculation would show [we have to use here the equation $z = z_0 + \rho e^{i\theta}$ for the circle $\gamma\,(0 \leqslant \theta \leqslant 2\pi)$], are equal to zero except the one which corresponds to $n = k$ and is equal to $2\pi i$. Consequently

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-z_0)^{k+1}}\,dz = a_k \qquad (k=0, \pm 1, \pm 2, \ldots). \tag{9}$$

We have thus obtained expressions for the coefficients of the Laurent series in terms of the sum of this series. It thus follows that if the sums of the Laurent series

$$f(z) = \sum_{-\infty}^{+\infty} a_k (z-z_0)^k \quad \text{and} \quad \varphi(z) = \sum_{-\infty}^{+\infty} b_k (z-z_0)^k,$$

converge in the circular rings D and Δ, containing one and the same circle $|z-z_0| = \rho$ and, further, coincide at all the points on the circle, then the corresponding coefficients of the two series will be equal :

$$a_k = b_k \qquad (k=0, \pm 1, \pm 2, \ldots),$$

i. e. the two series will be identical. In particular, the series will be identical if the rings D and Δ coincide and $f(z) = \varphi(z)$ at all points of the ring D. From what has been said above it follows that *the Laurent expansion possesses the uniqueness property.*

From the property of uniqueness, we find, exactly in the same way as in Art. 3 of Chapter 4, that if the expansion $\sum_{-\infty}^{+\infty} a_k z^k$ represents an even function, then the coefficients of odd powers of z in it are equal to zero; on the other hand, if the function is odd, then all the coefficients of even powers in it are equal to zero.

2. LAURENT'S THEOREM

We shall now prove the following important proposition.

Laurent's theorem. *Any function $f(z)$ which is one-valued and analytic in the circular ring $D: r < |z-z_0| < R$, can be represented in this ring by a convergent Laurent series*

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z-z_0)^n.$$

We note that under the conditions of the theorem the ring may degenerate into a circle, with its centre excluded ($r=0$, $R<\infty$), or into the exterior of circle, with the point at infinity excluded ($0<r$, $R=\infty$) or, finally, into the whole plane, with two points z_0 and ∞ excluded ($r=0$, $R=\infty$). All these cases are covered in the following proof.

Let z be any point in the ring D. We form a new ring D':

$$r' < |\zeta - z_0| < R',$$

lying in the interior of the first ring and containing the point z (Fig. 48). For this purpose it is sufficient to take

$$r < r' < |z - z_0| < R' < R.$$

Further, let $|\zeta - z| = \rho$ be a circle, with its centre at z, lying in the interior of D'. Since

$$\varphi(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

is an analytic function of ζ in the domain D, excluding the point $\xi = z$, hence, by the Cauchy integral theorem for a composite contour (Art. 9 of Chapter 5), we have

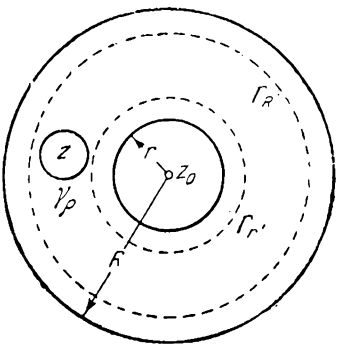


Fig. 48.

$$\frac{1}{2\pi i}\int_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{\zeta - z}\,; \tag{10}$$

here, $\Gamma_{R'}$, $\Gamma_{r'}$ and γ_ρ denote the circles

$$|\zeta - z_0| = R', \quad |\zeta - z_0| = r' \quad \text{and} \quad |\zeta - z_0| = \rho,$$

respectively, traversed during integration in the anti-clockwise direction. But the last integral in formula (10) is the Cauchy integral;

consequently, it is equal to $f(z)$. Hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{11}$$

We express $\frac{1}{\zeta - z}$ under the first integral ($\zeta \in \Gamma_{R'}$) in the form of the sum of a geometric series with common ratio $\frac{z - z_0}{\zeta - z_0}$, whose module is

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{R'} = \theta < 1.$$

We obtain

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_0^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{12}$$

Since for all the points ζ, belonging to $\Gamma_{R'}$, the module of the general term of the last series is

$$\left|\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\right| = \frac{1}{R'}\,\theta^n \qquad (0 < \theta < 1);$$

therefore, the series (12) converges uniformly on $\Gamma_{R'}$ (with respect to ζ). The series obtained from (12) by multiplying it by the function $\frac{1}{2\pi i}\,f(\zeta)$ (with module bounded on $\Gamma_{R'}$) will also converge uniformly:

$$\frac{1}{2\pi i}\,\frac{f(\zeta)}{\zeta - z} = \sum_0^\infty \frac{1}{2\pi i}\,\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,(\zeta - z_0)^n.$$

It thus follows that the last series can be integrated termwise on $\Gamma_{R'}$. We then obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_0^\infty a_n (z - z_0)^n, \tag{13}$$

where

$$a_n = \frac{1}{2\pi i}\int_{\Gamma_{R'}} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{14}$$

The first of the integrals on the right-hand side of (11) has thus been expanded into a convergent series in nonnegative powers of $z - z_0$.

Turning to the second integral on the right-hand side of (11), we represent $-\frac{1}{\zeta-z}$ $(\zeta \in \Gamma_{r'})$ as the sum of a geometric series with common ratio $\frac{\zeta-z_0}{z-z_0}$, whose module is

$$\left|\frac{\zeta-z_0}{z-z_0}\right| = \frac{r'}{|z-z_0|} = \vartheta < 1.$$

We obtain

$$-\frac{1}{\zeta-z} = \frac{1}{z-z_0}\frac{1}{1-\frac{\zeta-z_0}{z-z_0}} = \sum_0^\infty \frac{(\zeta-z_0)^n}{(z-z_0)^{n+1}}. \tag{15}$$

Observing that this series also converges uniformly on $\Gamma_{r'}$, we multiply all its terms by $\frac{1}{2\pi i} f(\zeta)$ and integrate termwise, with the result

$$-\frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{f(\zeta)}{\zeta-z}\,d\zeta = \sum_1^\infty a_{-n}(z-z_0)^{-n}, \tag{16}$$

where

$$a_{-n} = \frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{-n+1}} \quad (n=1, 2, \ldots). \tag{17}$$

Thus, the second integral on the right-hand side of (11) has been represented as a convergent series in negative powers of $(z-z_0)$.

Putting the expansions (13) and (16) derived above into the right-hand side of (11), we obtain an expansion of the function $f(z)$ into a Laurent series, for any arbitrary point $z \in D$:

$$f(z) = \sum_0^\infty a_n(z-z_0)^n + \sum_1^\infty a_{-n}(z-z_0)^{-n} = \sum_{-\infty}^{+\infty} a_n(z-z_0)^n. \tag{18}$$

The coefficients of this expansion are calculated partly by the formula (14) and partly by the formula (17). Taking any arbitrary circle $\Gamma : |z-z_0| = \lambda$, where $r < \lambda < R$, we find, by the Cauchy integral theorem for a composite contour, that they can be calculated by an integration along the circle Γ :

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}} \quad (n=0, \pm1, \pm2, \ldots). \tag{19}$$

This result is in agreement with the one obtained earlier [vide formula (9)].

3. ISOLATED SINGULAR POINTS OF A SIMPLE CHARACTER

We shall now examine the one-valued function $f(z)$ analytic in the neighbourhood of the point z_0, excluding perhaps the point itself. Then, $f(z)$ is analytic in a certain domain D:

$$0 < |z-z_0| < R.$$

Regarding the point z_0 we can make two assumptions. Firstly, it is possible that there exists a finite complex number a_0 such that, by putting $f(z_0) = a_0$, we obtain the function $f(z)$ analytic in the whole circle $|z-z_0| < R$ (including the point z_0). Secondly, it is possible that no such number exists. In the first case, we say that z_0 is a *regular point* of the function $f(z)$ or that the function $f(z)$ is *regular* at this point. In the second case, z_0 is said to be *an isolated singular point of a simple character* or, simply, a *singular point* of the function $f(z)$.

The basic means for the representation and study of an analytic function in the neighbourhood of an isolated singular point z_0 is the Laurent series. We shall apply Laurent's theorem to the function $f(z)$ in the domain $D: 0 < |z-z_0| < R$. This domain is a degenerated circular ring with its internal radius $r = 0$. We obtain

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z-z_0)^n \qquad (z \in D), \tag{20}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta \qquad (n=0, \pm 1, \pm 2, \ldots),$$

γ_ρ being a circle with its centre at the point z_0 and of radius ρ $(0<\rho<R)$. The expansion (20) can also be used when z_0 is a regular point. In this case, the Laurent series changes into a Taylor series and we have: $a_{-1} = a_{-2} = \ldots = 0$.

We shall now prove the following fundamental proposition.

Theorem. *In order that the function $f(z)$, one-valued and analytic in the domain $D: 0<|z-z_0| < R$, be regular at the point z_0, it is necessary and sufficient that there exists a neighbourhood U of the point z_0 in which $f(z)$ is bounded in its module.*

Proof. Let z_0 be a regular point of the function $f(z)$. Then $f(z)$ can be so defined at the point z_0 that it is analytic and, hence, bounded in

some neighbourhood of the point z_0. The necessity of the condition of the theorem is thereby proved.

We shall now prove its sufficiency. Let there exist a neighbourhood U of the point z_0 and a positive number $M < \infty$, such that

$$|f(z)| < M \quad \text{for all } z \in U.$$

Then, choosing ρ $(0<\rho<R)$ such that the circle γ_ρ belongs to U, we obtain from formula (19) the following estimates for the module of the coefficients a_n of the Laurent series (20) :

$$|a_n| < \frac{1}{2\pi} M \frac{2\pi\rho}{\rho^{n+1}}, \quad \text{i.e. } |a_n| < \frac{M}{\rho^n}.$$

We examine here only coefficients of the negative powers of $z-z_0$, i.e. for $n < 0$. Let ρ (subject only to the condition $0 < \rho < R$) tend to zero ; then

$$a_n = 0 \quad \text{for } n = -1, -2, -3, \ldots .$$

Thus, the Laurent series changes into a Taylor series

$$f(z) = \sum_0^\infty a_n (z-z_0)^n .$$

We have simply to put $f(z_0) = a_0$, to see that z_0 is a regular point of $f(z)$.

From the theorem proved above it follows that if the point z_0 is to be an isolated singular point of $f(z)$, it is necessary and sufficient that the module $|f(z)|$ be unbounded in any of its neighbourhoods, i.e. the condition

$$\overline{\lim_{z\to z_0}} |f(z)| = +\infty \tag{21}$$

has to be fulfilled. It thus follows that there are *a'priori* two possible modes of behaviour of $f(z)$ in the neighbourhood of an isolated singular point :

(*a*) $\lim_{z\to z_0} f(z) = \infty$;

(*b*) there does not exist any finite or infinite limit of the function $f(z)$ as z tends to z_0.

Either of these possibilities is realized in practice.

Let us take $f(z) = \frac{1}{(z-z_0)^n}$, where n is a natural number.

Obviously, this function is analytic for $0 < |z-z_0|$ and, for it, $\lim_{z \to z_0} f(z) = \infty$. Thus, the condition (*a*) holds.

As a second example, we take $f(z) = e^{\frac{1}{z-z_0}}$. This function is also analytic for $0 < |z-z_0|$ but, as distinguished from the previous one, the $\lim_{z \to z_0} f(z)$ does not exist. Indeed, if the point z lies on a straight line, passing through z_0, parallel to the real axis so that $z-z_0 = x-x_0$ is a real number, then, for $x > x_0$, as $x \to x_0$, $e^{\frac{1}{x-x_0}} \to \infty$ and, for $x < x_0$, as $x \to x_0$, $e^{\frac{1}{x-x_0}} \to 0$. Thus, in this case, the condition (*b*) holds. Fig. 49 shows the surface $u = \left| \exp \frac{1}{z} \right|$, which is a relief of the function in question (for $z_0 = 0$).[1]

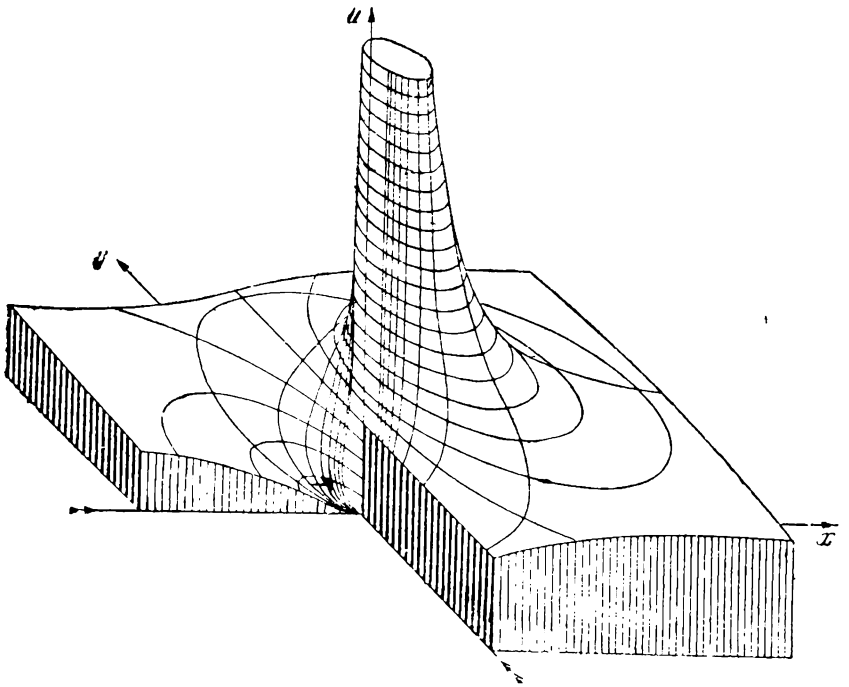


Fig. 49.

The isolated singular point z_0 of a simple character for which the condition (*a*), $\lim f(z) = \infty$, holds is called *a pole of the analytic function.*

The isolated singular point z_0 of a simple character for which the condition (*b*) holds, viz. the $\lim_{z \to z_0}$ (either finite or infinite) does not exist, is called *an essential singular point of the function.*

[1] This figure has been taken from the *Tables of Functions* by Jahnke and Emde.

We shall now study in detail each of the two types of singularities.

Let z_0 be a pole of the function $f(z)$. Then $\lim_{z \to z_0} |f(z)| = \infty$ and, consequently, there exists a neighbourhood $|z-z_0| < \delta < R$ of the point z_0 in which $f(z)$ satisfies the inequality $|f(z)| > 1$. The function $\varphi(z) = \frac{1}{f(z)}$ will, evidently, be analytic everywhere in this neighbourhood, except perhaps the point z_0. But since $\varphi(z) = \frac{1}{|f(z)|} < 1$, it follows by the theorem of this article that the point z_0 is a regular point of $\varphi(z)$. The value of this function at the point z_0 is equal to $\lim_{z \to z_0} \frac{1}{f(z)} = 0$. Hence, the point z_0 is a zero of the function $\varphi(z)$.

Conversely, if we know that $\varphi(z)$ is a function, one-valued and analytic in some neighbourhood of the point z_0, and that z_0 is a zero of this function, with the condition $\varphi(z) \not\equiv 0$, then we can find a $\Delta > 0$ so small that $\varphi(z)$ does not have in the circle $|z-z_0| < \Delta$ zeros other than the point z_0 (*cf.* Art. 7, Chapter 6). We form the function $f(z) = \frac{1}{\varphi(z)}$; it is one-valued and analytic for $0 < |z-z_0| < \Delta$ and tends to ∞, when z tends to z_0. Hence, z_0 is a pole of $f(z)$.

Thus, we have proved the following proposition. *In order that the point z_0 be a pole of the function $f(z)$, it is necessary and sufficient that it be a zero of the function* $\varphi(z) = \frac{1}{f(z)}$.

By virtue of this property, which establishes a correspondence between zeros and poles, it becomes possible to introduce the notion of *multiplicity* or *order of multiplicity* (in short, *order*) of a pole. We shall say that the point z_0 is a pole of multiplicity k ($k \geqslant 1$) of the function $f(z)$, if it is a zero of order k of the function $\frac{1}{\varphi(z)}$. When $k = 1$, the pole will be called a *simple pole* and when $k > 1$ a *multiple pole*.

In the neighbourhood of a pole of multiplicity k, the Laurent expansion has a definite character; we shall now explain this character. In fact, we shall prove the following proposition:

In order that the point z_0 be a pole of multiplicity k of the function $f(z)$, it is necessary and sufficient that the Laurent expansion of $f(z)$ in the neighbourhood of the point z_0 does not contain terms with powers lower than $-k$ and that the coefficient of $(z-z_0)^{-k}$ is nonzero. In other words, the Laurent expansion of the function $f(z)$ must have the form

$$f(z) = a_{-k}(z-z_0)^{-k} + \ldots + a_{-1}(z-z_0) + a_0 + a_1(z-z_0) + \ldots, \qquad (22)$$

where $a_{-k} \neq 0$.

Now, let z_0 be a pole of the function $f(z)$, of multiplicity k. Then, for $\frac{1}{f(z)}$ we must have at this point a zero of order k; hence

$$\frac{1}{f(z)} = A_k (z-z)^k + A_{k+1} (z-z_0)^{k+1} + \ldots \quad (A_k \neq 0)$$

in some neighbourhood of the point z_0. Thus

$$f(z) = \frac{1}{(z-z_0)^k} \cdot \frac{1}{A_k + A_{k+1}(z-z_0) + \ldots} \cdot \tag{23}$$

The power series $A_k + A_{k+1}(z-z_0) + \ldots$ represents an analytic function which does not vanish in any neighbourhood of the point z_0 (because $A_k \neq 0$). Hence, the function $\frac{1}{A_k + A_{k+1}(z-z_0) + \ldots}$ is analytic in the neighbourhood of z_0 and admits of an expansion of the form $\alpha_0 + \alpha_1 (z - z_0) + \ldots$, where $\alpha_0 = \frac{1}{A_k} \neq 0$. Putting the last series into the formula (23), we obtain for $f(z)$ the expansion

$$f(z) = \alpha_0 (z-z_0)^{-k} + \alpha_1 (z-z_0)^{-k+1} + \ldots \quad (\alpha_0 \neq 0),$$

which, by virtue of the uniqueness of expansion into a Laurent series, is a Laurent expansion of the function $f(z)$. It is the same as (22), with the notations $(\alpha_n = a_{n-k}, n = 0, 1, 2, \ldots)$.

Thus, the condition of the theorem to be proved is found to be necessary. We shall now show that it is also sufficient. Let $f(z)$ possess in the neighbourhood of z_0 an expansion of the form (22), with $a_{-k} \neq 0$. Rewriting this in the form

$$f(z) = \frac{a_{-k} + a_{-k+1}(z-z_0) + \ldots}{(z-z_0)^k},$$

we obtain

$$\frac{1}{f(z)} = (z - z_0)^k \frac{1}{a_{-k} + a_{-k+1}(z-z_0) + \ldots},$$

or, replacing the function $\frac{1}{a_{-k} + a_{-k+1}(z-z_0) + \ldots}$ by its Taylor expansion in powers of $z - z_0$, we get

$$\frac{1}{f(z)} = (z - z_0)^k [\beta_0 + \beta_1 (z-z_0) + \ldots] = \\ = \beta_0 (z-z_0)^k + \beta_1 (z-z_0)^{k+1} + \ldots,$$

where $\beta_0 = \frac{1}{a_{-k}} \neq 0$.

We have thus found that the point z_0 is a zero of order k of the function $\frac{1}{f(z)}$; the same point is a pole of multiplicity k of the function (z). Thus the theorem is proved.

4. THE THEOREM OF SOKHOTSKII

We now turn to the case of an essential singularity. The behaviour of a function in the neighbourhood of an essential singularity is characterized by the following proposition.

The theorem of Yu. V. Sokhotskii. *Whatever be the complex number A (proper or improper), there exists a sequence of points $\{z_n\}$, converging to the essential singularity z_0, such that* $\lim_{n\to\infty} f(z_n) = A$.

Proof. When $A = \infty$, the theorem is true because the function $f(z)$ is not bounded in its module in any neighbourhood of the essential singularity. Now, let $A \neq \infty$. We shall then prove the theorem indirectly. If, in any arbitrary neighbourhood of the point z_0, we cannot find points at which the values of the function are as close as desired to A, then there must exist a neighbourhood $0 < |z-z_0| < \delta$ and a number $\alpha > 0$, such that $|f(z)-A| > \alpha$ for $0 < |z-z_0| < \delta$. We now construct the function $\varphi(z) = \frac{1}{f(z)-A}$; it is analytic in the neighbourhood $0 < |z-z_0| < \delta$. Besides, it satisfies in this neighbourhood the inequality

$$|\varphi(z)| = \frac{1}{|f(z)-A|} < \frac{1}{\alpha}.$$

Hence, by the first theorem of Art. 3, $\varphi(z)$ is regular at the point z_0 and its value at this point must be equal to the limit : $\lim_{z\to z_0} \frac{1}{f(z)-A}$. But, $f(z)$ is not bounded in any neighbourhood of the point z_0. Hence, the above-mentioned limit can be only zero, i. e. $\varphi(z_0) = 0$. Thus, the function $\frac{1}{f(z)-A}$ has a zero at the point z_0, whence it follows that the function $f(z) - A$, and therefore also $f(z)$, has a pole at this point. We thus arrive at a contradiction with the condition of the theorem, whence it follows that the theorem is true.

We shall illustrate this theorem by two examples.

Example 1. $f(z) = \sin \frac{1}{z}$. Here, the origin of coordinates is an essential singularity. Indeed, as z tends to zero, $\sin \frac{1}{z}$ does not tend to any limit, finite or infinite, which follows immediately when we examine just a few of the real values of z.

If $A = \infty$, then assuming, $z_n = \frac{i}{n}$, and, say, consequently, $\frac{1}{z_n} = -in$, we obtain

$$\sin \frac{1}{z_n} = -i \sinh n \to \infty \quad \text{as } n \to \infty,$$

Now, let $A \neq \infty$. To obtain the sequence $\{z_n\}$, which appears in the theorem of Sokhotskii, we shall solve the equation

$$\sin \frac{1}{z} = A;$$

we obtain

$$\frac{1}{z} = \operatorname{Arcsin} A = \frac{1}{i} \operatorname{Ln} (iA + \sqrt{1-A^2}),$$

whence

$$z = \frac{i}{\operatorname{Ln} (iA+\sqrt{1-A^2})} = \frac{i}{\ln|iA+\sqrt{1-A^2}|+2k\pi i}.$$

Assuming

$$z_n = \frac{i}{\ln|iA+\sqrt{1-A^2}|+2n\pi i}$$

and ascribing to n the values 1, 2, 3, . . ., we obtain the sequence $\{z_n\}$, converging to zero and satisfying the conditions

$$f(z_n) = A \quad (n=1, 2, \ldots),$$

and, consequently, $\lim\limits_{n\to\infty} f(z_n) = A$.

Example 2. $f(z) = e^{\frac{1}{z}}$. Here, the origin of coordinates is an essential singularity because here again the limit $\lim\limits_{z\to 0} e^{\frac{1}{z}}$ does not exist.

Assuming $A = \infty$, we take $z_n = 1/n$. Now, $f(z_n) = e^n \to \infty$ for $n \to \infty$, i.e. the sequence $\{1/n\}$ is suited to the theorem of Sokhotskii for $A = \infty$. Now, let $A = 0$. Then, assuming $z_n = -1/n$, we have $f(z_n) = e^{-n}$ as $n \to \infty$, i.e. the theorem is true in this case too. Finally, let $A \neq 0$, $A \neq \infty$. In this case, it is most convenient to choose the corresponding points z_n by solving the equation

$$e^{\frac{1}{z}} = A.$$

We obtain

$$\frac{1}{z} = \operatorname{Ln} A,$$

and, hence,

$$z = \frac{1}{\operatorname{Ln} A} = \frac{1}{\ln|A| + 2k\pi i}.$$

Putting

$$z_n = \frac{1}{\ln|A| + 2n\pi i} \quad (n = 1, 2, \ldots),$$

we get a sequence $\{z_n\}$ which converges to zero and satisfies the condition $f(z_n) = A$. Thus, $\lim_{n\to\infty} f(z_n) = A$.

From the Sokhotskii theorem it follows that if z_0 is an essential singularity of the function $f(z)$ and E_δ a set of values taken by the function in any arbitrary small neighbourhood $|z - z_0| < \delta$ of this point, then the closure of the set E_δ (i.e. E_δ along with all the limit points of this set) coincides with the extended complex plane.

In fact, every point A of the complex plane is a limit of the sequence $\{f(z_n)\}$ of the points belonging to E_δ; hence, A belongs to the closure of the set E_δ.

In Examples 1 and 2 we have seen that, with isolated exceptions ($A = \infty$ in the first example, $A = \infty$ and $A = 0$ in the second), we can, instead of the sequence of the points $\{z_n\}$ for which

$$\lim_{z_n \to z_0} f(z_n) = A$$

is satisfied, derive such sequences for which the *exact equalities*

$$f(z_n) = A \quad (n = 1, 2, \ldots)$$

hold. A similar situation is also observed in the general case. This can be stated as follows :—

The theorem of Picard. *If z_0 is an essential singularity of the function $f(z)$, then, for every $A \neq \infty$, save at most one value $A = A_0$, there exists an infinite sequence of A-points of the function $f(z)$ which converges to z_0.*[1]

In the example $f(z) = \sin 1/z$, there is no exceptional value and in the case $f(z) = e^{\frac{1}{z}}$ this one is zero, because the function $e^{\frac{1}{z}}$ is always nonzero. It can be easily verified that the Sokhotskii theorem is contained in the statement of the Picard theorem.

From the latter theorem it follows that the set of values of the function $f(z)$, which it takes up in any arbitrary neighbourhood $|z - z_0| < \delta$

[1] See, in this connection, A.I. Markushevich, loc. cit., Chapter 8, Art. 8.4.

of the essential singularity z_0 coincides with the whole finite plane $|z| < \infty$, save at most one point A_0 (A_0 not depending on δ).

The Laurent expansion of the function $f(z)$ in the neighbourhood of the essential singularity z_0 must contain an infinity of terms with negative powers of $z-z_0$ (of course, with nonzero coefficients). Indeed, if such terms were completely absent in this expansion, the point z_0 would be a proper point of $f(z)$ and if there were a finite number of such terms, the point z_0 would be a pole of $f(z)$ (according to the theorem of Art. 3). Conversely, *whenever a Laurent expansion of the function $f(z)$ in the neighbourhood of a certain point z_0 contains an infinity of terms with negative powers of $z - z_0$, z_0 is an essential singularity of the function $f(z)$.* Indeed, it can neither be a regular point of $f(z)$ (because in that case the terms with negative powers must be completly absent) nor a pole (because in that case there must be only a finite number of such terms).

As an example, we consider the function $\exp 1/z^4$. For its validity we have the expansion, convergent for any $z \neq 0$,

$$\exp \frac{1}{z^4} = 1 + \frac{1}{z^4} + \frac{1}{2!\,z^8} + \frac{1}{3!\,z^{12}} + \dots .$$

Evidently, it can be looked upon as a Laurent expansion in the neighbourhood of the point $z = 0$. Since this expansion contains an infinity of terms with negative powers of z, hence $z = 0$ is an essential singularity of the function. Obviously, this can be established if we observe the behaviour of the function in the neighbourhood of the origin of coordinates. The reader can easily see that it (the function) tends to ∞ as z approaches the origin of coordinates along the coordinate axes and to zero as z approaches the origin of coordinates along the bisectors of the coordinate angles. Hence, $\lim\limits_{z\to 0} \exp 1/z^4$ does not exist, and the point $z = 0$ is an essential singularity of the function $\exp 1/z^4$.

From what has been said above, it follows that the totality of terms with negative powers in the Laurent expansion of the function $f(z)$ in the neighbourhood of a singularity determines the character of the singularity. For this reason the series $\sum\limits_{1}^{\infty} a_{-k}(z-z_0)^{-k}$ is called *the principal part of the Laurent expansion* $\sum\limits_{-\infty}^{+\infty} a_k(z-z_0)$ in the neighbourhood of the point z_0. The series $\sum\limits_{0}^{\infty} a_k(z-z_0)^k$, consisting of all the terms

of the expansion with nonnegative powers represents a function which is regular at the point z_0 and, hence, is called the *regular part of the Laurent series.*

While applying the propositions mentioned above, we should remember that they refer only to those Laurent expansions which converge in some neighbourhood $0 < |z-z_0| < R$ of the point under question.

As an example, we shall consider the Laurent series

$$\ldots + \frac{1}{z^n} + \frac{1}{z^{n-1}} + \ldots + \frac{1}{z} + \frac{1}{2} + \frac{z}{2^2} + \ldots + \frac{z^n}{2^{n+1}} + \ldots .$$

It contains an infinity of terms with negative powers of z. However, before we say that $z = 0$ is an essential singularity for the sum of this series we must see whether it converges in any neighbourhood of this point. We note that the given series represents the sum of two progressions $\sum_1^\infty z^{-n}$ and $\sum_0^\infty \frac{z^n}{2^{n+1}}$. The first of these converges for $|z| > 1$ and represents the function $\frac{1/z}{1-1/z} = \frac{1}{z-1}$. The second converges for $|z| < 2$ and represents the function $\frac{1/2}{1-z/2} = \frac{1}{2-z}$. Hence, the domain of convergence of the given series is the ring $1 < |z| < 2$, which, of course, is not a neighbourhood of the origin of coordinates. The sum of the series in this ring is equal to $\frac{1}{z-1} + \frac{1}{2-z} = \frac{-1}{z^2-3z+2}$, a function for which the origin of coordinates is a regular point and all the singularities it has are the two simple poles : $z = 1$ and $z = 2$.

5. SINGULARITIES OF THE DERIVATIVES AND OF THE RATIONAL COMBINATIONS OF ANALYTIC FUNCTIONS

For a quick determination of the position and nature of the singularities of a function in definite cases it is useful to keep in view the following simple propositions derived from the theorems of Arts. 3 and 4.

(*a*) If $f(z)$ and $\varphi(z) \not\equiv 0$ are two functions, one-valued and analytic in a given domain G, then the function $F(z) = \frac{f(z)}{\varphi(z)}$ can have, in the domain G, singularities, namely poles, only at the zeros of the function $\varphi(z)$. Let ζ be a k-tuple zero of the function $\varphi(z)$ ($k \geqslant 1$) and an l-tuple zero of the function $f(z)$ ($l \geqslant 0$) (when ζ is not a zero of the function $f(z)$, we take $l = 0$). Then, in the neighbourhood of the point ζ, we have

$$F(z) = \frac{\frac{f^{(l)}(\zeta)}{l!}(z-\zeta)^l + \ldots}{\frac{\varphi^{(k)}(\zeta)}{k!}(z-\zeta)^k + \ldots} = (z-\zeta)^{l-k} \frac{\frac{f^{(l)}(\zeta)}{l!} + \ldots}{\frac{\varphi^{(k)}(\zeta)}{k!} + \ldots},$$

where $f^{(l)}(\zeta) \neq 0$ and $\varphi^{(k)}(\zeta) \neq 0$. It follows that $F(z)$ has at the point ζ a pole of order $k - l$ if $k > l$ and a regular point if $k \leqslant l$, being a zero of the function $F(z)$ of order $l - k$ if $k < l$.

(*b*) If $f(z)$ and $\varphi(z)$ are two functions not having in the domain G singularities other than poles, then their sum, difference, product and quotient (this last is formed only when $\varphi(z) \not\equiv 0$) also do not have singularities other than poles.

Let us examine, in particular, the difference $f(z) - \varphi(z)$ of these functions and let ζ be the point in the neighbourhood of which the Laurent expansions of the functions $f(z)$ and $\varphi(z)$ are

$$f(z) = \frac{a_{-l}}{(z-\zeta)^l} + \ldots + \frac{a_{-1}}{z-\zeta} + a_0 + a_1(z-\zeta) + \ldots$$

and

$$\varphi(z) = \frac{b_{-k}}{(z-\zeta)^k} + \ldots + \frac{b_{-1}}{z-\zeta} + b_0 + b_1(z-\zeta) + \ldots .$$

Here, l and k denote respectively the orders of the point ζ when looked upon as a pole of the two functions. For greater generality, we shall assume that $l \leqslant 0$ (or $k \leqslant 0$) when ζ is a regular point of $f(z)$ [or of $\varphi(z)$] and develop in this case an expansion in the negative powers of $z - \zeta$.

Subtracting the expansion for $f(z)$ from the expansion for $\varphi(z)$ term by term, we obtain an expansion for $f(z) - \varphi(z)$. Obviously, the point ζ will be a pole for this difference if, and only if, it is a pole at least for one of the functions $f(z)$ and $\varphi(z)$ ($k \geqslant 1$ or $l \geqslant 1$) and the principal parts of the expansions for $f(z)$ and $\varphi(z)$ do not coincide with each other. When the principal parts coincide (i.e. $k = l$, $a_{-k} = b_{-k}, \ldots,$ $, \ldots, a_{-1} = b_{-1}$), we obtain for the difference the expansion

$$f(z) - \varphi(z) = a_0 - b_0 + (a_1 - b_1)(z-\zeta) + \ldots ,$$

and it follows that ζ is then a regular point of $f(z) - \varphi(z)$.

We further examine the function

$$F(z) = \frac{f(z)}{\varphi(z)} \qquad [\varphi(z) \not\equiv 0].$$

Its singularities, if any, lie only at the zeros of the function $\varphi(z)$ or at the poles of the function $f(z)$. Let ζ be a point which is either a zero or a pole of the function $f(z)$ or $\varphi(z)$. In all cases we can write

$$f(z) = (z-\zeta)^l [a_0 + a_1(z-\zeta) + \ldots]$$

and

$$\varphi(z) = (z-\zeta)^k [b_0 + b_1(z-\zeta) + \ldots],$$

where l and k are integers (positive, negative or zero) and the square brackets enclose power series converging in some neighbourhood of the point ζ and possessing constant nonzero terms ($a_0 \neq 0$, $b_0 \neq 0$). Here, for $k > 0$, $\varphi(z)$ has a k-tuple zero at the point ζ when $k = 0$, $z = \zeta$ is a regular point at which $\varphi(z) \neq 0$ and, finally, when $k < 0$, $z = \zeta$ is a singularity of the function $\varphi(z)$, that is, a pole of order $-k$.

We shall substitute the expansions for $f(z)$ and $\varphi(z)$ in the formula for $F(z)$. Thus

$$F(z) = (z-\zeta)^{l-k} \frac{a_0 + a_1(z-\zeta) + \dots}{b_0 + b_1(z-\zeta) + \dots}.$$

Evidently, when $l \geqslant k$ the point ζ will be regular for $F(z)$ (in particular, when $l > k$, it will be a zero of order $l-k$), and when $l < k$, it will be a pole of $F(z)$ [of order $k - l$].

(*c*) Let $f(z)$ be a one-valued function, not having in the domain G singularities other than poles. Then, the derivative $f'(z)$ of this function also cannot have in the domain G singularities other than poles. In fact, $f'(z)$ has a pole at every pole of the function $f(z)$ but the multiplicity of a pole of $f'(z)$ is greater by one than the multiplicity of the corresponding pole of $f(z)$.

Now, let ζ be a pole of the function $f(z)$, with multiplicity $k \geqslant l$. Then, $f(z)$ has, in some neighbourhood U of the point ζ, $0 < |z-\zeta| < R$, the expansion

$$f(z) = \frac{A_{-k}}{(z-\zeta)^k} + \dots + \frac{A_{-1}}{z-\zeta} + A_0 + A_1(z-\zeta) + \dots \quad (A_{-k} \neq 0).$$

Since the terms of this expansion are analytic in U and the expansion itself converges uniformly in the interior of U (by the property of a Laurent series), it can be differentiated term by term in U. We thus obtain

$$f'(z) = -\frac{kA_{-k}}{(z-\zeta)_{k+1}} - \dots - \frac{A_{-1}}{(z-\zeta)^3} + A_1 + \dots \quad (kA_{-k} \neq 0).$$

This is the Laurent expansion of $f'(z)$ in the neighbourhood of the point ζ and from this we can see that ζ is a pole of multiplicity $k+1$ of the derivative $f'(z)$.

(*d*) Let $f(z) \not\equiv \text{const}$ be a one-valued function, not having singularities other than poles in the domain G, and let $A \neq \infty$ be an arbitrary complex number. Then, the *logarithmic derivative* of the function $f(z) - A$,

$$\frac{d\{\operatorname{Ln}[f(z)-A]\}}{dz} = \frac{f'(z)}{f(z)-A},$$

does not possess in the domain G singularities other than poles. In fact, it has simple poles at all the poles of the function $f(z)$ and at all the A-points of this function [i.e. at all the zeros of the function $f(z)-A$].

For verifying this statement we go over to the general case examined above under (*b*). We found that the quotient of two functions may have poles at the zeros of the denominator or at the poles of the numerator. Let the point $z=\zeta$ be a k-tuple zero of the denominator, i.e. an A-point of the function $f(z)$ of multiplicity k. Then

$$f(z)-A=a_0(z-\zeta)^k+a_1(z-\zeta)^{k+1}+\ldots \qquad (k\geqslant 1,\quad a_0\neq 0).$$

It follows that

$$f'(z)=ka_0(z-\zeta)^{k-1}+a_1(k+1)(z-\zeta)^k+\ldots,$$

i. e. the point ζ is a $(k-1)$-tuple zero of the numerator of the fraction. Hence, this point is a simple pole of the logarithmic derivative.

Let, on the other hand, $z=\zeta$ be a pole of the numerator $f'(z)$. This is possible only when ζ is a pole of $f(z)-A$. In this case, as we have seen in (*c*), the multiplicity of the pole of $f'(z)$ will be higher by unity than the multiplicity of the same pole for $f(z)-A$. Consequently, for the logarithmic derivative we again obtain a simple pole at the point ζ.

(*e*) If ζ is a regular point or a pole of the function $f(z)\not\equiv 0$ and is an essential singularity of the function $\varphi(z)$, then ζ will also be an essential singularity of each of the functions $\varphi(z)\pm f(z)$, $f(z)\varphi(z)$ and $\dfrac{\varphi(z)}{f(z)}$.

We denote the latter functions by $\psi_1(z)$, $\psi_2(z)$, and $\psi_3(z)$, respectively. Then we have

$$\varphi(z)=\psi_1 z\pm f(z),\quad \varphi(z)=\frac{\psi_2(z)}{f(z)},\ \varphi(z)=\psi_3(z)f(z).$$

If we assume that $\psi_j(z)$ $(j=1, 2, 3)$ have a regular point or a pole at $z=\zeta$, then the function $\varphi(z)$ will also have a regular point or a pole at $z=\zeta$, which contradicts our assumption. Thus, the point $z=\zeta$ cannot be a regular point of the functions $\psi_j(z)$. Since these functions are one-valued and analytic in some neighbourhood of the point ζ, excluding this point, this must be an isolated singularity of a simple character for $\psi_j(z)$. But we have already seen that the point ζ cannot be a pole of $\psi_j(z)$. Hence, it is an essential singularity of each of these functions.

(*f*) If ζ be an essential singularity of $\varphi(z)$, then the function $\dfrac{1}{\varphi(z)}$ will have at ζ either an essential singularity or a *nonisolated singularity which is a limit point of the poles.*

Now, there are two possibilities. Either there exists a neighbourhood of the point ζ in which $\varphi(z)$ does not vanish or there does not exist such a neighbourhood. In the first case, the function $\psi(z) = \frac{1}{\varphi(z)}$ will be analytic in some neighbourhood of the point ζ, excluding the point ζ itself. This point can be neither a regular point nor a pole of $\psi(z)$, otherwise ζ would be a pole or a regular point of $\varphi(z) = \frac{1}{\psi(z)}$, which goes against our assumption. Hence, ζ is an essential singularity of $\psi(z)$.

In the second case, in every neighbourhood of the point ζ there would be zeros of the function $\varphi(z)$ and, consequently, there would be, in this neighbourhood, poles of the function $\psi(z) = \frac{1}{\varphi(z)}$. From this it follows that any neighbourhood of the point ζ contains singularities (that is, poles) of the function $\psi(z)$. Hence, in the case under consideration, ζ is a nonisolated singularity of $\psi(z)$. And this is the limit point of the poles.

6. THE CASE OF THE POINT AT INFINITY

We examine here a one-valued function $f(z)$, analytic at all points of the exterior, $|z| > r$, of some circle with its centre at the origin of coordinates, excluding possibly the point at infinity. Performing the transformation $z = \frac{1}{\zeta}$, we reduce the study of such a function to the study of the function $f^*(\zeta) = f\left(\frac{1}{\zeta}\right)$, which is analytic at all points in the neighbourhood of the origin of coordinates, excluding possibly the origin of coordinates itself. In this case, the point $\zeta = 0$ will be an image of the point at infinity $z = \infty$ and, to every sequence of points $\{z_n\}$, converging to the point at infinity, there will correspond a sequence of points $\left\{\zeta_n = \frac{1}{z_n}\right\}$ converging to zero, and conversely.

Depending on whether the point $\zeta = 0$ is a regular point, a pole of order k or an essential singularity of $f^*(\zeta)$, we shall call the point $z = \infty$ a regular point, a pole of order k or an essential singularity. Since in the cases mentioned above $f^*(\zeta)$ will have, in the neighbourhood of the point $\zeta = 0$, Laurent expansions

$$f^*(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n + \ldots,$$

$$f^*(\zeta) = a_k\zeta^{-k} + \ldots + a_1\zeta^{-1} + a_0 + a_{-1}\zeta + \ldots \qquad (a_k \neq 0)$$

and

$$f^*(\zeta) = \sum_{-\infty}^{+\infty} a_n\zeta^{-n},$$

respectively (where in the last case the infinite set of coefficients a_n of the negative powers of ζ are nonzero), hence *the function* $f(z) = f^*\left(\frac{1}{z}\right)$, *in the neighbourhood of the point at infinity, must have Laurent expansions of the form*

$$f(z) = a_0 + a_{-1} z^{-1} + a_{-2} z^{-2} + \ldots + a_{-n} z^{-n} + \ldots,$$

$$f(z) = a_k z^k + \ldots + a_1 z + a_0 + a_{-1} z^{-1} + \ldots \quad (a_k \neq 0)$$

and

$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n,$$

depending respectively on whether this point is a regular point, a pole of order k or an essential singularity (in the last case the infinite set of coefficients of the positive powers of z will be nonzero).

Thus, the connection between the character of the point (at infinity) in relation to a function and its corresponding expansion as a Laurent series is the same as in the case of a finite point, with the only difference that the roles of the terms with positive and negative powers get interchanged. Accordingly, the principal part of the Laurent expansion in the neighbourhood of the point at infinity is the totality of terms with positive powers, while the regular part is the totality of terms with non-positive powers.

We know that it is possible to distinguish between a regular point, a pole and an essential singularity at $\zeta = 0$ without examining the corresponding Laurent expansions, and simply by studying the realisation of any of the three possibilities, viz. that

1) $f^*(\zeta)$ is bounded in the neighbourhood of the origin of coordinates ;

2) the limit of $f^*(\zeta)$, as ζ tends to zero, is equal to infinity ;

3) there exists neither a finite nor an infinite limit of $f^*(\zeta)$, as ζ tends to zero.

Since $f(z) = f^*(\zeta)$ and $z = \frac{1}{\zeta}$, it follows that the same criteria hold also for the point at infinity, and the function $f(z)$ will have a regular point, a pole or an essential singularity at $z = \infty$, depending respectively on whether this function is bounded in a neighbourhood of the point at infinity or it tends to infinity as $z \to \infty$ or, finally, whether it does not possess any limit, finite or infinite, as $z \to \infty$.

7. ENTIRE AND MEROMORPHIC FUNCTIONS

An entire function $f(z)$, by its very definition, being one-valued and analytic in the *whole* of the finite plane, does not have singularities in the

finite plane. Hence, it can have a singularity only at the point at infinity. Since $f(z)$ can be represented by the power series

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots,$$

everywhere convergent, the series represents the function in any neighbourhood of the point at infinity; consequently, it coincides with the Laurent expansion of the function in the neighbourhood of the point at infinity. Obviously, when $z = \infty$ is a regular point of $f(z)$, $a_1 = a_2 = \ldots$ $\ldots a_n = \ldots = 0$ and $f(z) \equiv a_0$. This result follows from the theorem of Liouville also. From the fact that the point $z = \infty$ is a regular point of $f(z)$, it follows that $f(z)$ is bounded in its module in some neighbourhood of this point and, consequently, bounded in the whole plane; hence, $f(z) \equiv$ const.

When the point $z = \infty$ is a pole of $f(z)$ of order k, we have

$$a_k \neq 0,\ a_{k+1} = a_{k+2} = \ldots = 0,$$

whence

$$f(z) \equiv a_0 + a_1 z + \ldots + a_k z^k,$$

that is, $f(z)$ is a polynomial (an entire rational function) of degree k.

Finally, when $z = \infty$ is an essential singularity of $f(z)$, there must be among the coefficients of $f(z)$ an infinity of nonzero numbers. In this case, $f(z)$ is different from a polynomial and is called an *entire transcendental function.* Examples of such functions are e^z, $\sin z$, $\cos z$, ...

Thus, *an entire function can have at the infinity a regular point, a pole or an essential singularity. The entire function will correspondingly be a constant, an entire polynomial (the degree of which is the same as the order of the pole) or an entire transcendental function.*

A function *meromorphic* in the finite plane (derived from the Greek words, *meros*: part, fraction, and *morphos*: form, type) is a function which can be represented in the form of a quotient of two entire functions:

$$f(z) = \frac{g(z)}{h(z)} \qquad [h(z) \not\equiv 0].$$

By Art. 5, we conclude that this function cannot have at the finite points of the plane singularities other than poles. In fact, $f(z)$ has a pole at the point z_0 if, and only if, this point is a zero of the denominator $[h(z_0) = 0]$ and, moreover, the numerator is either nonzero at this point, $[g(z_0) \neq 0]$, or has a zero of an order lower than that of the zero of the denominator. In this case, the order of the pole z_0 is equal to the difference

between the order of the zero of the denominator and that of the zero of the numerator. If there exists in the whole finite plane only a finite number of poles of $f(z)$, then there exists a neighbourhood of the point $z = \infty$, such that no finite singularity of $f(z)$ lies in it. Hence, in this case, as in the case of an entire function, $f(z)$ can have at infinity either a regular point or a pole or an essential singularity. Let $z_1, z_2, \ldots, z_m$ be the various possible poles of $f(z)$ and let $\beta_1, \beta_2, \ldots, \beta_m$ be their multiplicities. Then, in the neighbourhood of the point z_j, $f(z)$ has a Laurent expansion of the form

$$f(z) = \frac{A_{-\beta_j}^{(j)}}{(z-z_j)^{\beta_j}} + \ldots + \frac{A_{-1}^{(j)}}{z-z_j} + A_0^{(j)} + A_1^{(j)}(z-z_0) + \ldots .$$

The principal part of this expansion is the rational function $G_j(z)$, possessing in the whole of the extended plane only one singularity—a pole of order β_j at the point z_j. This function vanishes at the point $z = \infty$. Evidently, the difference $f(z) - G_j(z)$ does not have a singularity at $z = z_j$. The remaining points $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$ are the poles of this function. The principal parts of the Laurent expansions for $f(z) - G_j(z)$ are the same as the corresponding principal parts for the function $f(z)$. In fact, $G_j(z)$ is regular at each of the points z_k $(k \neq j)$; consequently, its expansion in the neighbourhood of z_k contains only nonnegative powers of $z - z_k$, thus not affecting the principal parts of the expansions. If from $f(z)$ we subtract the sum of the principal parts, $\sum_{j=1}^{m} G_j(z)$, we obtain the function $\varphi(z) = f(z) - \sum_{j=1}^{m} G_j(z)$, which has no singularity in the finite plane and is, therefore, an entire function. Since $\lim_{z\to\infty} [f(z) - \varphi(z)] = \lim_{z\to\infty} \sum_{j=1}^{m} G_j(z) = 0$, the functions $f(z)$ and $\varphi(z)$ behave similarly at the point at infinity, i.e. they possess simultaneously either a regular point or a pole or an essential singularity there; in the last two cases, the principal parts of the Laurent expansions of $f(z)$ and $\varphi(z)$ are equal in the neighbourhood of the point $z = \infty$. Thus, in the case under examination, the meromorphic function $f(z)$ can be represented in the form of a sum of the entire function $\varphi(z)$ and the rational function $\sum_{j=1}^{m} G_j(z)$:

$$f(z) = \varphi(z) + \sum_{j=1}^{m} G_j(z).$$

When $f(z)$ has either a regular point or a pole of order k at the point $z=\infty$, the entire function $\varphi(z)$ will be, respectively, either a constant: $\varphi(z) \equiv \equiv \lim_{z\to\infty} f(z) = A_0$, or a polynomial of degree k : $A_0 + A_1 z + \ldots + A_k z^k$, where $A_1 z + \ldots + A_k z^k$ is the principal part of the expansion of $f(z)$ in the neighbourhood of the point $z = \infty$. Denoting this principal part by $G_0(z)$ and taking $G_0(z) \equiv 0$ if the point at infinity is a regular point of $f(z)$, we obtain

$$f(z) = A_0 + \sum_{j=0}^{m} G_j(z). \tag{24}$$

This result may be formulated in terms of the following theorem: *If a meromorphic function $f(z)$ has only a finite number of poles in the finite plane and the point at infinity is either a regular point or a pole for it, then $f(z)$ is a rational function; moreover, $f(z)$ can be represented as a sum of the principal parts $G_j(z)$ of its Laurent expansions at all the finite poles ($j=1, 2, \ldots, m$) and at the point at infinity ($j=0$), and the constant*

$$A_0 = \lim_{z\to\infty} [f(z) - \sum_{0}^{m} G_j(z)].$$

We note that this theorem establishes in passing the existence of the expansion of any rational function into *partial fractions*. In fact, rewriting the formula (24) in an expanded form, we get

$$f(z) = A_0 + A_1 z + \ldots + A_k z^k + \sum_{j=1}^{n} \left[\frac{A_{-\beta_j}^{(j)}}{(z-z_j)^{\beta_j}} + \ldots + \frac{A_{-1}^{(j)}}{z-z_j}\right],$$

and this is the expansion of $f(z)$ into partial fractions.

Finally, we turn to the case when the set of the poles of the meromorphic function $f(z)$ is infinite. It can be easily seen that any closed circle $|z| \leqslant R < \infty$ can contain only a finite number of poles. Assuming the contrary, we find that our circle contains at least one point ζ which is a limit point for the poles. But such a point ζ can neither be a regular point nor a pole of $f(z)$ because there does not exist any neighbourhood of this point in which the function $f(z)$ can be analytic everywhere, except possibly at the point ζ itself. The assumption of the existence of such a (finite) point for a meromorphic function is not possible.

Thus, every circle $|z| \leqslant R$ contains in it only a finite number of poles of $f(z)$ and so the neighbourhood $|z| > R$ of the point at infinity contains, for any R, an infinite set of poles of $f(z)$. It follows that, in the given

case, the point at infinity is a limit point of the poles, i.e. it is a nonisolated singularity. We shall now show that *all the poles of $f(z)$ may be numbered and arranged in a sequence in the order of nondecreasing moduli.* For this, we divide the whole plane, by circles with their centres at the origin of coordinates, into zones K_0 ($|z| \leqslant 1$), K_1 ($1 < |z| \leqslant 2$), K_2 ($2 < |z| \leqslant 3$), ... Each of these zones, contains only a finite number (in special cases, zero) of poles of the function $f(z)$. Hence, numbering the different poles in the order of nondecreasing moduli,

$$0 \leqslant |z_0| \leqslant |z_1| \leqslant \ldots \leqslant |z_{n_j}|,$$

lying in the closed circle $|z| \leqslant j$, we can continue the numbering of the poles that lie in the zone K_j adjacent to the circle :

$$|z_{nj}| < |z_{nj+1}| \leqslant \ldots \leqslant |z_{nj+1}|.$$

Continuing this process without limit, we obtain a sequence $\{z_n\}$ in which all the poles of $f(z)$ are included, with the condition $|z_n| \leqslant |z_{n+1}|$.

Let $G_j(z)$ denote the principal part of the Laurent expansion of $f(z)$ in the neighbourhood of z_j :

$$G_j(z) = \frac{A^{(j)}_{-\beta_j}}{(z-z_j)^{\beta_j}} + \frac{A^{(j)}_{-\beta_j+1}}{(z-z_j)^{\beta_j-1}} + \ldots + \frac{A^{(j)}_{-1}}{(z-z_j)^{-1}}.$$

In the given case, we can proceed as in the case of a meromorphic function with a finite number of poles, i.e. we subtract the sum of all the principal parts from $f(z)$ so that in the difference we obtain a function which does not have singularities in the finite plane ; in other words, it is an entire function. However, in the given case, the set of the principal parts is infinite and there is no guarantee that the series $\sum_{j=0}^{\infty} G_j(z)$ will converge. The Swedish mathematician Mittag-Leffler removed this difficulty, by showing that it is always possible to choose polynomials $P_j(z)$, such that the series $\sum_{j=0}^{\infty} [G_j(z) - P_j(z)]$ will uniformly converge in any circle $|z| < R$; if we exclude from this circle the points $z_0, z_1, \ldots$, the poles of the various terms of the series will fall into the circle. Assuming that such polynomials are already found, we notice that $F(z) = \sum_{j=0}^{\infty} [G_j(z) - P_j(z)]$ represents a meromorphic function having poles at those points z_j at which $f(z)$ also has poles. In this case, the principal parts of the Laurent expansions $f(z)$ and $F(z)$, in the neighbourhood of z_j, are one and

the same rational functions $G_j(z)$. Hence, the difference $f(z) - F(z) = \varphi(z)$ is an entire function and we obtain the formula

$$f(z) = \varphi(z) + \sum_{j=0}^{\infty} [G_j(z) - P_j(z)] =$$

$$= \varphi(z) + \sum_{j=0}^{\infty} \left[\frac{A_{-\beta_j}^{(j)}}{(z-z_j)^{\beta_j}} + \ldots + \frac{A_{-1}^{(j)}}{z-z_j} - (C_0^{(j)} + \ldots + C_{\alpha_j}^{(j)} z^{\alpha_j}) \right]. \quad (25)$$

This is the so-called *Mittag-Leffler expansion* of the meromorphic function $f(z)$. It can also be regarded as an *expansion* of a meromorphic function into *partial fractions*. Without giving a proof of the Mittag-Leffler theorem, we give here a simple example of an expansion which will be proved later (vide Art. 5, Chapter 8)

$$\cot z = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z-z_j} + \frac{1}{z_j} \right) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right].$$

Here, $z_0 = 0$, $z_{2j-1} = j\pi$, $z_{2j} = -j\pi$ $(j = 1, 2, 3, \ldots)$, $\frac{1}{z-z_j}$ is the principal part $G_j(z)$ of the Laurent expansion of $\cot z$ in the neighbourhood of the point z_j and $1/z_j$ is the simplest polynomial $P_j(z)$ (in the given case, of zero degree) which has to be added to $G_j(z)$ to ensure its convergence. Finally, the entire function $\varphi(z)$ in the general formula (25) is identically equal to zero in the given case.

8. EXPANSION OF AN ENTIRE FUNCTION INTO A PRODUCT

In this article we shall show that for any entire function $f(z)$, having zeros, there exists an expansion into factors similar to the expansion of a polynomial into factors. To emphasize the analogy we rewrite the expansion of a polynomial into factors in a form somewhat different from the usual one. Let $P(z)$ be a polynomial, $z_1, \ldots, z_n$ be its zeros, different from the origin of coordinates (among them some may be equal; these correspond to multiple roots), and finally, let $z = 0$ be a zero of $P(z)$ with multiplicity λ [if $P(0) \neq 0$, we take $\lambda = 0$]. Then, $P(z)$ can be represented in the form of the following product:

$$P(z) = Cz^{\lambda} \left(1 - \frac{z}{z_1}\right) \ldots \left(1 - \frac{z}{z_n}\right) = Cz^{\lambda} \prod_{1}^{n} \left(1 - \frac{z}{z_j}\right).$$

For the sake of comparison, we take the entire function $\sin z$, having simple zeros : $0, \pm \pi, \pm 2\pi, \ldots, \pm n\pi, \ldots$. If we arrange the zeros of $\sin z$, which are different from the origin of coordinates, in the order of nondecreasing moduli, putting $z_{2j-1} = j\pi$, $z_{2j} = -j\pi$, then we can prove the formula

$$\sin z = z \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right). \tag{26}$$

This formula is similar in all respects to the expansion of a polynomial. We shall prove this in Art. 5 (Chapter 8).

In the general case, however, we do not obtain a formula so simple as this. In fact, if $f(z)$ is an entire function with the zeros : 0 (with multiplicity λ), $z_1, z_2, \ldots, z_n, \ldots$ ($|z_n| \leqslant |z_{n+1}|$), then the product $z^{\lambda} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right)$ may diverge. To remove this difficulty, Weierstrass introduced in the product certain additional factors of the form $e^{\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}}$, which never vanish by themselves but ensure the convergence of the infinite product. If the numbers k_j are so chosen that the product $z^{\lambda} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}}$ converges uniformly in any closed circle $|z| \leqslant R$, it represents an entire function $F(z)$, which has the same zeros as the function $f(z)$. Hence, we conclude that either $f(z)$ coincides with $F(z)$ or differs from it by a factor of the form $e^{g(z)}$ where $g(z)$ is also an entire function (this factor vanishes nowhere). Finally, we obtain for $f(z)$ the Weierstrass formula :

$$f(z) = e^{g(z)} z^{\lambda} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}}. \tag{27}$$

Let us now prove this formula.

Lemma. *If the entire functions $f(z) \not\equiv 0$ and $F(z) \not\equiv 0$ have one and the same zeros (with the same multiplicities), then*

$$f(z) = e^{g(z)} F(z),$$

where $g(z)$ is some entire function (constant, in a special case).

In fact, the quotient $\varphi(z) = \frac{f(z)}{F(z)}$ can have singularities (poles) only at the zeros of $F(z)$. But, any zero of $F(z)$ is a zero of $f(z)$, of the same multiplicity, hence $\varphi(z)$ does not have any pole in the finite plane and is, consequently, an entire function. The arguments with the help of which we established that $\varphi(z)$ does not have poles also show that $\varphi(z)$ does not have any zeros either. Hence, $h(z) = \frac{\varphi'(z)}{\varphi(z)}$ is an entire function (Art. 5, *a*). Integrating $h(z)$ from zero to an arbitrary value z, we also obtain the entire function

$$g_1(z) = \int_0^z h(z)\,dz = \int_0^z \frac{\varphi'(z)}{\varphi(z)}\,dz = \ln \frac{\varphi(z)}{\varphi(0)},$$

whence

$$\varphi(z) = \varphi(0)\, e^{g_1(z)} = e^{g_1(z)+\ln \varphi(0)} = e^{g(z)},$$

where $g(z) = g_1(z) + \ln \varphi(0)$ is also an entire function.

Thus,

$$f(z) = \varphi(z)\, F(z) = e^{g(z)}\, F(z).$$

For proving this lemma we shall establish that any entire function $\varphi(z)$ having no zero can be represented in the form $\varphi(z) = e^{g(z)}$, where $g(z)$ is also an entire function.

Let $f(z)$ be an entire function, having a finite number of zeros: $\underbrace{0, \ldots, 0}_{\lambda}, z_1, z_2, \ldots, z_n$. Since the polynomial $F(z) = z^\lambda \left(1 - \frac{z}{z_1}\right) \ldots \left(1 - \frac{z}{z_n}\right)$ has the same zeros, we have by the lemma

$$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right).$$

To obtain a similar expression for any entire function, having an infinite number of zeros, we have to first construct at least one entire function having the given zeros. We now show that such a function always exists.

Theorem. *For any sequence* $\{z_n\}$, *converging to* ∞, *of complex numbers different from the origin of coordinates and arranged in the order of non-decreasing moduli* ($|z_n| \leqslant |z_{n+1}|$), *we can construct an entire function* $f(z)$ *whose zeros coincide with* z_n.

Note. Among the members of the sequence we may come across some which are equal to one another (arranged one after the other). If $z_{n_0+1} = z_{n_0+2} = \ldots = z_{n_0+\alpha} = a$, and the other members of the sequence are different from a, then it means that the number a must be a zero of $f(z)$ of multiplicity α.

Proof. Let $\{k_n\}$ be a sequence of nonnegative integers, such that the series $\sum_{n=1}^{\infty} \left(\frac{R}{|z_n|}\right)^{k_n+1}$ converges for any $R \geqslant 0$. For any sequence $\{z_n\}$, the numbers $k_n = [\ln n]$, ($\ln n$) being the integral part of $\ln n$, are suitable as the required k_n. Indeed, if $\frac{|z_n|}{R} > e^2$ for $n > N_0$, then $\left(\frac{|z_n|}{R}\right)^{k_n+1} > e^{2([\ln n]+1)} > e^{2 \ln n} = n^2$, and hence $\left(\frac{R}{|z_n|}\right)^{k_n+1} < \frac{1}{n^2}$. But, in the remaining cases, numbers equal to one another can be chosen for the k_n. For instance, if $|z_n| = n$, then we can put $k_n = 1$; if $z_n = n^2$, then it is sufficient to take $k_n = 0$ ($n = 1, 2, \ldots$). We shall show that the unknown entire function can be represented as an infinite product:

$$\prod_{1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}}$$

(if $k_j = 0$, then the corresponding power index has to be set equal to zero). Let R be any arbitrary positive number. We denote by $N(R)$ a natural number such that $|z_n| > 2R$ if $n > N(R)$. Taking $n > N(R)$, we represent the partial product $\prod_{1}^{n}$ as

$$\prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}} = \prod_{j=1}^{N(R)} \cdot \prod_{N(R)+1}^{n} \cdot \qquad (28)$$

The first product represents an entire function which has the given zeros $z_1, z_2, \ldots, z_{N(R)}$. Hence, the zeros of the product, in the circle $|z| \leqslant 2R$, satisfy the conditions of the theorem. We write the second product in the form

$$\prod_{N(R)+1}^{n} = \prod_{N(R)+1}^{n} \exp\left\{\ln\left(1 - \frac{z}{z_j}\right) + \frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}\right\} =$$

$$= \exp \sum_{N(R)+1}^{n} \left\{\ln\left(1 - \frac{z}{z_j}\right) + \frac{z}{z_j} + \ldots + \frac{z^{k_j}}{k_j z_j^{k_j}}\right\}.$$

Since $|z| \leqslant R$ and $|z_j| > 2R$, we have $\left|\frac{z}{z_j}\right| < \frac{1}{2}$; hence,

$$\left|\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right| =$$

$$= \left|\frac{z^{k_j+1}}{(k_j+1)\,z_j^{k_j+1}}+\frac{z^{k_j+2}}{(k_j+2)\,z_j^{k_j+2}}+\ldots\right| \leqslant$$

$$\leqslant \frac{|z|^{k_j+1}}{|z_j|^{k_j+1}}+\frac{|z|^{k_j+2}}{|z_j|^{k_j+2}}+\ldots \leqslant \frac{R^{k_j+1}}{|z_j|^{k_j+1}}\left(1+\frac{R}{|z_j|}+\frac{R^2}{|z_j|^2}+\ldots\right) =$$

$$= \frac{R^{k_j+1}}{|z_j|^{k_j+1}}\,\frac{1}{1-\frac{R}{|z_j|}} < \frac{2R^{k_j+1}}{|z_j|^{k_j+1}}\,[j > N(R)].$$

From the inequality obtained above and the convergence of the series $\sum_1^{\infty}\left(\frac{R}{|z_j|}\right)^{k_j+1}$, it follows that the series $\sum_{N(R)+1}^{\infty}\left\{\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\right.$ $\left.+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right\}$ converges absolutely and uniformly in the circle $|z| < R$. Hence, the sum of the series $\varphi_R(z)$ is one-valued and analytic in this circle.

Going over to the limit $n \to \infty$ in the relation (28), we obtain

$$\lim_{n\to\infty}\prod_{j=1}^{n}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}} = \prod_{j=1}^{\infty} = \prod_{j=1}^{N(R)}\cdot e^{\varphi_R(z)}. \qquad (29)$$

This limit is a function, analytic in the circle $|z| < R$, having specified zeros in this circle. Since this result is applicable to a circle of any radius R, however large, the infinite product $\prod_{j=1}^{\infty}$ converges, in the whole of the finite plane, to a function which is analytic everywhere, i.e. to an entire function having the specified zeros $\{z_n\}$:

$$f(z) = \prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}. \qquad (30)$$

Example 1. $z_n = n^2$. Here, we can put $k_n = 0$. We obtain the simplest entire function with these zeros as

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{n^2}\right).$$

Example 2. $z_n = n$. Here we can put $k_n = 1$. Thus we obtain

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.$$

Example 3. Let the sequence $\{z_n\}$ be 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ... Here again $k_n = 1$, and the corresponding infinite product can be written in the form

$$f(z) = (1-z)\, e^z \left[\left(1 - \frac{z}{2}\right) e^{\frac{z}{2}}\right]^2 \cdot \left[\left(1 - \frac{z}{3}\right) e^{\frac{z}{3}}\right]^3 \cdots .$$

Now, let $f(z) \not\equiv 0$ be an arbitrary entire function, having an infinite number of zeros. Noting that every closed circle $|z| \leqslant R < \infty$ can contain only a finite number of zeros of this function, we find, as in the case of poles of a meromorphic function (Art. 7), that all these zeros can be arranged in a sequence in the order of nondecreasing moduli :

$$\underbrace{0, 0, \ldots, 0}_{\lambda}, z_1, z_2, \ldots, z_n, \ldots, \lim z_n = \infty.$$

If $k_n \geqslant 0$ are integers such that the series $\sum_1^\infty \left(\frac{R}{|z_n|}\right)^{k_n+1}$ converges for any R, then, by the theorem proved above, the function

$$F(z) = z^\lambda \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{z^{k_n}}{k_n z_n^{k_n}}}$$

is an entire function, having the same zeros as the function $f(z)$. Hence

$$f(z) = e^{g(z)} z^\lambda \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{z^{k_n}}{k_n z_n^{k_n}}},$$

where $g(z)$ is some entire function (which can be equal to zero). This is the Weierstrass formula.

As an illustration, we take the function $\frac{1}{\Gamma(z)}$. Here, $\Gamma(z)$ is the gamma function[1], a meromorphic function with which the reader is already acquainted through a general course of analysis where it is treated as a function of a real positive argument. We can show that the function $\Gamma(z)$ has no zeros and that it has simple poles at the points $0, -1, -2, \ldots, -n, \ldots$, whence it follows that $\frac{1}{\Gamma(z)}$ is an entire function, with simple zeros at the above-mentioned points. Putting $k_n = 1$ $(n=1, 2, 3, \ldots)$ and noting that $\lambda = 1$ here (λ is the multiplicity of the point $z = 0$, regarded as a zero of the entire function), we get

$$\frac{1}{\Gamma(z)} = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

The entire function $g(z)$ can be defined by making use of the special properties of the gamma function. On finds that $g(z) = Cz$, where C is the so-called *Euler constant*:

$$C = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right) = 0.5772 \ldots .$$

Thus,

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{n=1} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

The fundamental theorem of this article enables us to prove that any function $\Phi(z)$, one-valued and analytic in the finite plane, excepting those points where it has poles, is a meromorphic function, i.e. it can be represented in the form of a quotient of two entire functions. Now we construct an entire function $F(z)$, having zeros at those points where $\Phi(z)$ has poles, the multiplicity of each zero being equal to the multiplicity of the corresponding pole. Then, the product $F(z)\,\Phi(z) = H(z)$ will, as can be easily verified, be an entire function; consequently, the function

$$\Phi(z) = \frac{H(z)}{F(z)}$$

is a meromorphic function.

Hence follows the identity of the two classes of analytic functions: *(a) functions that do not possess in the finite part of the plane singularities*

[1] For a detailed treatment, see. A.I. Markushevich, loc. cit., Chapter 7, Art. 4.

other than poles, and (b) functions that can be represented in the form of a quotient of two entire functions. Indeed, we have just established that every function of the class (*a*) is contained in the class (*b*). Conversely, a function of class (*b*) cannot have finite singularities other than poles (Art. 7); hence it follows that our statement is true. Now, we can give the following definition of a *meromorphic function* (being equivalent to the previous definition): a one-valued analytic function $f(z)$ is called *meromorphic* if it does not have, in the finite plane, singularities other than poles.

9. THE ORDER AND THE TYPE OF AN ENTIRE FUNCTION

Here, we shall introduce the concept of the *order* and the *type* of an entire function $f(z)$—the two numerical characteristics which give an idea of the rate of increase of the maximum module of the function: $M(r) = \max_{|z|=r} |f(z)|$, as $r \to \infty$. By virtue of the Liouville theorem, $\lim_{r\to\infty} M(r) = \infty$ if $f(z) \not\equiv \text{const}$. We shall now compare the increase of $\ln \ln M(r)$ with $\ln r$. We call the number

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln \ln M(r)}{\ln r} \tag{31}$$

the *order* of an entire function.

If $\rho = +\infty$, then the function $f(z)$ is called *a function of infinite order.* As an example of a function of infinite order we may take $f(z) = e^{e^z}$. Here, $M(r) = e^{e^r}$, $\ln \ln M(r) = r$ and $\rho = \lim_{r\to\infty} \frac{r}{\ln r} = \infty$.

If $\rho < +\infty$ $(\rho \geqslant 0)$, then $f(z)$ is called *a function of finite order.* Since, for any $\varepsilon > 0$, there exists $R_0(\varepsilon) > 0$, such that

$$\frac{\ln \ln M(r)}{\ln r} < \rho + \varepsilon \quad \text{for } r > R_0(\varepsilon),$$

we have

$$M(r) < e^{r^{\rho+\varepsilon}} \quad \text{for } r > R_0(\varepsilon). \tag{32}$$

On the other hand, for any $\varepsilon > 0$, there exists an increasing sequence of positive numbers $\{r_n(\varepsilon)\}$ $(\lim r_n = \infty)$ such that

$$\frac{\ln \ln M(r_n)}{\ln r_n} > \rho - \varepsilon,$$

whence we have

$$M[r_n(\varepsilon)] > e^{r_n(\varepsilon)^{\rho-\varepsilon}}. \tag{33}$$

A comparison of the inequalities (32) and (33) shows that the order ρ of an entire function can be defined as the lower bound of those non-negative numbers α, for which the inequality

$$M(r) < e^{r^\alpha} \quad \text{for} \quad r > R_1(\alpha)$$

holds.

Example 1. If $f(z) = e^{z^n}$ (n being a natural number), then $M(r) = e^{r^n}$; hence, $\rho = \lim\limits_{r\to\infty} \dfrac{\ln \ln M(r)}{\ln r} = n$. Thus, e^{z^n} is a function of order n and, in particular, e is a function of order one.

Example 2. Let $f(z) = \sin z$. Since $|\sinh y| \leqslant |\sin z| \leqslant \cosh y$ (*cf.* Art. 13, Chapter 3), so $\sinh r \leqslant M(r) \leqslant \cosh r$. Consequently, $\rho = 1$, i.e. the order of $\sin z$ is equal to unity. Exactly in the same way, we can prove that the order of $\cos z$ is also equal to unity.

Example 3. Let $f(z) = \dfrac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \dfrac{z}{3!} + \dfrac{z^2}{5!} - \ldots$. Here, $\dfrac{\sinh \sqrt{r}}{\sqrt{r}} \leqslant M(r) \leqslant \dfrac{\cosh \sqrt{r}}{\sqrt{r}}$ and, consequently, $\rho = \dfrac{1}{2}$.

If the order ρ of the function $f(z)$ is a finite positive number, $0 < \rho < +\infty$, then in addition to ρ we can still have another numerical characteristic which could define more accurately the increase of the maximum module of an entire function. Let us compare $\ln M(r)$ with r^ρ, putting

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\ln M(r)}{r^\rho}; \tag{34}$$

σ defines the *type* of the entire function. If $\sigma = \infty$, then it is said that the given function of order ρ is a function of *infinite* or the *maximal* type. If $\sigma < \infty$ ($\sigma \geqslant 0$), then the function $f(z)$ is called a *function of finite type* or, more precisely, of *normal* or *average type* if $\sigma > 0$ and of *minimal type* if $\sigma = 0$.

Arguing in the same manner as in the discussion of the order of a function, we find that for any $\varepsilon > 0$ the following inequalities are satisfied:

$$M(r) < e^{(\sigma+\varepsilon)r^\rho}, \quad r > R(\varepsilon),$$

$$M(r_n) > e^{(\sigma-\varepsilon)r_n^\rho}, \quad r_n < r_{n+1}, \quad \lim_{n\to\infty} r_n = \infty.$$

Hence, it follows that the type of an entire function can be defined

as the lower bound of the numbers β for which the inequality

$$M(r) < e^{\beta r^{\rho}} \quad \text{for } r > R_2(\beta)$$

holds.

Example 1. Each of the functions (of the first order) e^{Az}, $\sin Az$, $\cos Az$ $(A \neq 0)$ is of the type characterized by σ: $\sigma = |A|$;

Example 2. The function $\exp(c_0 + c_1 z + \ldots + c_n z^n)$ $[n \geqslant 1,\ c_n \neq 0]$ has an order n and is characterized by the type $|c_n|$.

8

RESIDUES AND THEIR APPLICATIONS. THE PRINCIPLE OF THE ARGUMENT

1. THE THEOREM OF RESIDUES AND ITS APPLICATION TO THE EVALUATION OF DEFINITE INTEGRALS

We shall now deal with the evaluation of integrals of one-valued analytic functions along a closed contour, assuming that singularities other than the isolated singularities of a simple character are not included in the domain containing the contour of integration. When the problem is so stated, the interior of the contour can contain only a finite number of singularities (if not, the singularity might have at least one limit point which is also a singularity of the function but a nonisolated one). Let $z_1, z_2, \ldots, z_n$ be isolated singularities of the function $f(z)$ located in the interior of a rectifiable closed curve Γ. We draw around each of the points z_k a circle $\gamma_k : |z-z_k| = \rho_k$, of radius ρ_k, as small as desired, so that, this circle lies in the interior of Γ in such a way that each of these circles lies in the exterior of the others. Then, by the integral theorem for the composite contour, we have

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz.$$

Thus, the problem reduces to the evaluation of the integral $\int_{\gamma_k} f(z)\,dz$ along the circle $|z-z_k| = \rho_k$ which lies in the neighbourhood of the isolated singularity z_k of the function $f(z)$.

Replacing $f(z)$ by its Laurent expansion in the neighbourhood of the point z_k and integrating it termwise (which is possible because of the uniform convergence of the Laurent series on γ_k), we obtain

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} \sum_{m=-\infty}^{+\infty} a_m^{(k)} (z-z_k)^m\,dz = \sum_{m=-\infty}^{+\infty} a_m^{(k)} \int_{\gamma_k} (z-z_k)^m\,dz = a_{-1}^{(k)} 2\pi i.$$

In fact, of all the integrals $\int_{\gamma_k} (z-z_k)^m \, dz$, there is only one which is nonzero; this corresponds to the value $m = -1$, in which case the integral is equal to $2\pi i$.

Thus

$$\int_{\Gamma} f(z)\, dz = 2\pi i \,(a_{-1}^{(1)} + a_{-1}^{(2)} + \ldots + a_{-1}^{(n)}). \tag{1}$$

This formula fully solves the given problem. We find that the value of the integral of an analytic function depends, under the given assumptions, only on the coefficients of the power 'minus one' in the Laurent expansions of the function in the neighbourhoods of the singularities. These coefficients are called the *residues of the function.*

Thus, the *coefficient of* $(z-a)^{-1}$ *in the Laurent expansion of the function in the neighbourhood of a is called the residue of the function* $f(z)$ *at the isolated singularity* a.[1]

Formula (1) expresses the following theorem.

The theorem of residues. *The integral of the function* $f(z)$ *taken along a closed contour* Γ *which lies in a domain where the function is one-valued and analytic (excluding the isolated singularities of a simple character) and does not pass through the singularities, is equal to* $2\pi i$ *multiplied by the sum of the residues of the function at all singularities included within the interior of* Γ.

To apply this theorem we have to calculate the residues. These can be calculated without any difficulty in case the singularity of the function is a pole. First let a be a simple pole of the function. Then, we have the expansion

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1 (z-a) + \ldots$$

[1] The concept of residue is due to Cauchy. He has also given numerous applications of this concept to various problems of analysis. The term 'residue' follows from the fact that Cauchy arrived at this concept while searching for the *difference* between integrals taken along two paths having common termini, in between which certain poles of the function are *present.* Residues in this sense can be found in the "*Memoirs on Definite Integrals*" (1814). The term residue itself is first met with in Cauchy's article '*On a New Kind of Calculus Analogous to the Calculus of Infinitesimals*' which was included in the first volume of the "*Exercices de Mathematique* (1826). Cauchy has here introduced this concept with the words, 'Once the values of x which turn $f(x)$ to infinity have been found out, we add to one of these values, denoted by x_1, an infinitesimal quantity ε and expand $f(x_1+\varepsilon)$ into increasing powers of this quantity, then the first terms of the expansion will contain negative powers of ε and one of them will be the product of $1/\varepsilon$ with a finite coefficient. We call this coefficient the *residue* of the function $f(x)$ with respect to the particular value x_1 of the variable x'. After this article, Cauchy wrote a large number of other articles which were included in the three succeeding volumes of the "Exercices" (1826-1829). In these, he discussed the application of this theory to the evaluation of integrals, to the expansion of functions into series and into infinite products, to the theory of equations, etc.

in some neighbourhood of the point a. From this, we get

$$f(z)(z-a) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + \ldots$$

and, consequently,

$$a_{-1} = \operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} [f(z)(z-a)]. \tag{2}$$

The calculation of a residue is further simplified if $f(z)$ is of the form

$$f(z) = \frac{\varphi(z)}{\psi(z)},$$

where $\varphi(a) \neq 0$ and $\psi(z)$ has a simple zero at $z = a$ [i.e. $\psi(a) = 0$ and $\psi'(a) \neq 0$]. Then, $z = a$ is a simple pole of the function $f(z)$, and, by formula (2), we get

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Res}_{z=a} \frac{\varphi(z)}{\psi(z)} = \lim_{z \to a} \frac{\varphi(z)(z-a)}{\psi(z)} = \lim_{z \to a} \frac{\varphi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\varphi(a)}{\psi'(a)}. \tag{3}$$

When a is a pole of order k ($k>1$), we have in the neighbourhood of the point a the expansion

$$f(z) = \frac{a_{-k}}{(z-a)^k} + \ldots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \ldots .$$

From this, one gets

$$f(z)(z-a)^k = a_{-k} + a_{-k+1}(z-a) + \ldots + a_{-1}(z-a)^{k-1} + \ldots .$$

Differentiating termwise $k-1$ times, we obtain

$$\frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}} = (k-1)!\, a_{-1} + k(k-1)\ldots 2a_0(z-a) + \ldots$$

and, finally, as $z \to a$

$$(k-1)!\, a_{-1} = \lim_{z \to a} \frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}}$$

or

$$a_{-1} = \operatorname*{Res}_{z=a} f(z) = \frac{1}{(k-1)!} \lim_{z \to a} \frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}}. \tag{4}$$

Example 1. Evaluate the integral $\int_{-\infty}^{+\infty} F(x)\,dx$, where $F(x)$ is the

rational function : $F(x) = \dfrac{P(x)}{Q(x)}$, which does not possess any pole on the real axis and is such that the degree of the denominator $Q(x)$ exceeds at least by two units the degree of the numerator $P(x)$.

We take the contour of integration as shown in Fig. 50, where BCA is a semi-circle of radius R, with its centre at the origin of coordinates. We choose the radius R as large as desired, such that all the poles of the function $F(z)$ that are found in the upper half-plane are included in its interior. In this case, we have

$$\int_{-R}^{+R} F(x)\,dx + \int_{BCA} F(z)\,dz = 2\pi i \sum \operatorname{Res} F(z),$$

where the summation is extended over all the poles of the function $F(z)$ lying in the upper half-plane.

Since

$$|F(z)| = \left|\frac{P(z)}{Q(z)}\right| = \left|\frac{a_m z^m + \ldots + a_0}{b_n z^n + \ldots + b_0}\right| = \left|\frac{a_m}{b_n z^{n-m}}\right| \left|\frac{1 + \ldots + \dfrac{a_0}{a_m z^m}}{1 + \ldots + \dfrac{b_0}{b_n z^n}}\right|$$

and $n - m \geqslant 2$, we have, for sufficiently large values of $|z| = R$,

$$|F(z)| < \frac{2|a_m|}{|b_n| R^2} = \frac{C}{R^2}.$$

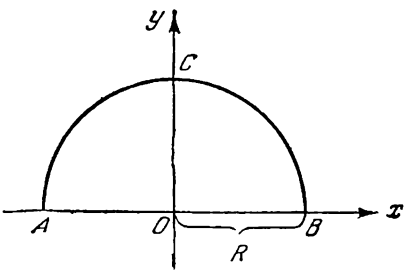


Fig. 50.

Hence

$$\left|\int_{BCA} F(z)\,dz\right| < \frac{C}{R^2}\pi R = \frac{\pi C}{R} \to 0$$

for $R \to \infty$.

Consequently,

$$\int_{-\infty}^{+\infty} F(x)\,dx = \lim_{R\to\infty} \int_{-R}^{+R} F(x)\,dx =$$

$$= 2\pi i \sum \operatorname{Res} F(z).$$

Thus, the integral of a rational function, not possessing poles on the real axis and possessing a zero at least of the second order (this condition is equivalent to the condition $n - m \geqslant 2$) at the point at infinity, is equal to $2\pi i$ multiplied by the sum of the residues of the function $F(z)$ at the poles that lie in the upper half-plane.

As a special case, let

$$F(z) = \frac{z^{2p}-z^{2q}}{1-z^{2r}},$$

where p, q and r are nonnegative integers subject to the conditions: $p < r$ and $q < r$.

Here, the degree of the denominator, $2r$, is higher by at least two units than the degree of the numerator. All the poles of the function $F(z)$ are given by the formula

$$z = e^{\frac{k\pi i}{r}} \quad (k=1, \ldots, r-1, r+1, \ldots, 2r-1)$$

[the points $+1$ and -1 are not poles of the function $F(z)$, because the numerator and the denominator of the fraction have a common factor $1 - z^2$; if $p - q$ and r are not mutually prime, then some of the above-mentioned points are no longer poles of $F(z)$]. Of these, the poles

$$z = e^{\frac{k\pi i}{r}} \quad (k=1, 2, \ldots, r-1)$$

lie in the upper half-plane; all these poles are simple and, hence,

$$\operatorname*{Res}_{z=e^{\frac{k\pi i}{r}}} F(z) = \frac{e^{\frac{k\pi i}{r}2p} - e^{\frac{k\pi i}{r}2q}}{-2re^{\frac{k\pi i}{r}(2r-1)}} = \frac{1}{2r}\left[e^{(2q+1)\frac{k\pi i}{r}} - e^{(2p+1)\frac{k\pi i}{r}}\right].$$

Thus

$$\int_{-\infty}^{+\infty} \frac{x^{2p}-x^{2q}}{1-x^{2r}}\,dx = \frac{\pi i}{r}\sum_{1}^{r-1}\left[e^{(2q+1)\frac{k\pi i}{r}} - e^{(2p+1)\frac{k\pi i}{r}}\right] =$$

$$= \frac{\pi}{r}\left[i\,\frac{1+e^{(2q+1)\frac{\pi i}{r}}}{1-e^{(2q+1)\frac{\pi i}{r}}} - i\,\frac{1+e^{(2p+1)\frac{\pi i}{r}}}{1-e^{(2p+1)\frac{\pi i}{r}}}\right] = \frac{\pi}{r}\left(\cot\frac{2p+1}{2r}\pi - \cot\frac{2q+1}{2r}\pi\right).$$

Noting that the function under the integral sign is an even function, we get

$$\int_0^{\infty} \frac{x^{2p}-x^{2q}}{1-x^{2r}}\,dx = \frac{\pi}{2r}\left(\cot\frac{2p+1}{2r}\pi - \cot\frac{2q+1}{2r}\pi\right).$$

If $r = 2n$ and $q = p + n\,(p<n)$, then the last formula assumes the form

$$\int_0^\infty \frac{x^{2p}}{1+x^{2n}}\,dx = \frac{\pi}{2n\sin\dfrac{2p+1}{2n}\pi}.$$

Example 2. Let $\Phi(z)$ be a function which is analytic, except at a finite number of poles in a certain domain containing the closed upper half-plane (excluding the point at infinity). If it does not have poles on the real axis and tends to zero as z approaches ∞ in the upper half-plane, then the formula

$$\int_0^\infty [e^{\mu ix}\Phi(x)+e^{-\mu ix}\Phi(-x)]\,dx = 2\pi i \sum \operatorname{Res}[e^{\mu iz}\Phi(z)]$$

holds for any $\mu > 0$; in this case, the summation is extended over all the poles of the function $\Phi(z)$ which lie in the upper half-plane.

Taking the same contour of integration as in Example 1, we get

$$\int_{-R}^{R} e^{\mu ix}\Phi(x)\,dx + \int_{BCA} e^{\mu iz}\Phi(z)\,dz = 2\pi i \sum \operatorname{Res}[e^{\mu iz}\Phi(z)].$$

We choose the radius R as large as desired, so that all the poles of the function $e^{\mu iz}\Phi(z)$, lying in the upper half-plane [these coincide with the poles of the function $\Phi(z)$], lie in the interior of the contour of integration.

We now show that, under the given conditions,

$$\lim_{R\to\infty} \int_{BCA} e^{\mu iz}\Phi(z)\,dz = 0.$$

In fact,

$$|J_R| = \left|\int_{BCA} e^{\mu iz}\Phi(z)\,dz\right| = \left|\int_0^\pi \exp(\mu iR\cos\varphi - \mu R\sin\varphi)\,\Phi(Re^{i\varphi})\,iRe^{i\varphi}\,d\varphi\right| \leqslant$$

$$\leqslant \int_0^\pi e^{-\mu R\sin\varphi}\,|\,\Phi(Re^{i\varphi})\,|\,R\,d\varphi.$$

By assumption, $\max\limits_{0\leqslant\varphi\leqslant\pi} |\Phi(Re^{i\varphi})| = \varepsilon(R) \to 0$ as $R \to \infty$. Hence

$$|J_R| < \varepsilon(R)\int_0^{\pi} Re^{-\mu R\sin\varphi}\,d\varphi = 2\varepsilon(R)\int_0^{\frac{\pi}{2}} Re^{-\mu R\sin\varphi}\,d\varphi <$$

$$< 2\varepsilon(R)\int_0^{\frac{\pi}{2}} Re^{-\frac{2}{\pi}\mu R\varphi}\,d\varphi = \frac{\pi\varepsilon(R)}{\mu}(1-e^{-\mu R}) < \frac{\pi\varepsilon(R)}{\mu} \to 0 \quad \text{as } R \to \infty.$$

Consequently

$$\lim_{R\to\infty}\int_{-R}^{+R} e^{\mu ix}\Phi(x)\,dx = \lim_{R\to\infty}\int_0^R [e^{\mu ix}\Phi(x) + e^{-\mu ix}\Phi(-x)]\,dx =$$

$$= \int_0^{\infty} [e^{\mu ix}\Phi(x) + e^{-\mu ix}\Phi(-x)]\,dx = 2\pi i\sum \operatorname{Res}[e^{\mu iz}\Phi(z)].$$

This is the required result. If $\Phi(z)$ is an even function, then the formula assumes the form

$$\int_0^{\infty} \cos\mu x\Phi(x)\,dx = \pi i\sum \operatorname{Res}[e^{\mu iz}\Phi(z)].$$

If $\Phi(z)$ is an odd function, then we obtain

$$\int_0^{\infty} \sin\mu x\Phi(x)\,dx = \pi\sum \operatorname{Res}[e^{\mu iz}\Phi(z)].$$

For instance,

$$\int_0^{\infty} \frac{\cos\mu x}{a^2+x^2}\,dx = \pi i\operatorname*{Res}_{z=ai}\frac{e^{\mu iz}}{a^2+z^2} = \frac{\pi e^{-\mu a}}{2a}$$

$$\int_0^{\infty} \frac{x\sin\mu x}{a^2+x^2}\,dx = \pi\operatorname*{Res}_{z=ai}\frac{ze^{\mu iz}}{a^2+z^2} = \frac{\pi e^{-\mu a}}{2}.$$

2. THE PRINCIPLE OF THE ARGUMENT AND ITS CONSEQUENCES

By way of one of the most important applications of the theory of residues, we derive the value of the integral $\frac{1}{2\pi i}\int_{\Gamma} \varphi(z)\frac{f'(z)}{f(z)-A}dz$, where $f(z)$ is a function which is one-valued in the domain G and does not possess in this domain singularities except certain poles, A is an arbitrary complex number, $\varphi(z)$ is a function one-valued and analytic in the same domain, and Γ is a close rectifiable Jordan curve lying, along with its interior, in the domain G and not passing either through the poles or the A-points of the function $f(z)$.

The singularities of the function $F(z) = \varphi(z)\frac{f'(z)}{f(z)-A}$ in the domain G can be only poles, arising from the A-points or the poles of the function $f(z)$. Let $a_1, \ldots, a_m$ be the A-points of $f(z)$, lying in the interior of Γ, $\alpha_1, \ldots, \alpha_m$ be the multiplicities of these points, $b_1, \ldots, b_n$ be the poles of the function $f(z)$ lying in the interior of Γ and $\beta_1, \ldots, \beta_n$ be the multiplicities of these poles. The functions $\varphi(z)$ and $f(z)$ have the expansions

$$\varphi(z) = \varphi(a_j) + \ldots, \quad f(z) - A = \alpha_j (z-a_j)^{\alpha_j} + \ldots$$

in the neighbourhood of the point a_j. Consequently

$$f'(z) = c_{\alpha_j}\alpha_j (z-a_j)^{\alpha_j - 1} + \ldots \text{ and}$$

$$F(z) = [\varphi(a_j) + \ldots]\frac{c_{\alpha_j}\alpha_j (z-a_j)^{\alpha_j-1} + \ldots}{c_{\alpha_j}(z-a_j)^{\alpha_j} + \ldots} =$$

$$= \frac{\alpha_j}{z-a_j}[\varphi(a_j) + \ldots]\frac{1+\ldots}{1+\ldots} = \frac{\alpha_j}{z-a_j}[\varphi(a_j) + \ldots] =$$

$$= \frac{\alpha_j \varphi(a_j)}{z-a_j} + \ldots .$$

The terms that are left out here contain higher powers of $z - a_j$. In particular, the term containing $(z-a_j)^{-1}$ must be followed by the constant term of the Laurent expansion, which in turn would be followed by the term containing $(z-a_j)$, and so on. Hence, it follows that $z = a_j$ is a simple pole of the function $F(z)$, having $\alpha_j\varphi(a_j)$ as the residue. This residue will be equal to zero if $\varphi(a_j) = 0$. In this latter case, the point a_j will not actually be a pole of the function $F(z)$.

We further examine any of the poles b_j of the function $f(z)$. In its neighbourhood, we have the expansions

$$\varphi(z) = \varphi(b_j) + \ldots,$$

$$f(z) - A = d_{-\beta_j}(z-b_j)^{-\beta_j} + \ldots$$

$$f'(z) = -\beta_j d_{-\beta_j}(z-b_j)^{-\beta_j-1} - \ldots,$$

and, hence,

$$F(z) = [\varphi(b_j) + \ldots] \frac{-\beta_j d_{-\beta_j}(z-b_j)^{-\beta_j-1} - \ldots}{d_{-\beta_j}(z-b_j)^{-\beta_j} + \ldots} =$$

$$= \frac{-\beta_j}{z-b_j}[\varphi(b_j) + \ldots] \frac{1+\ldots}{1+\ldots} = -\frac{\beta_j}{z-b_j}[\varphi(b_j)+\ldots] =$$

$$= -\frac{\beta_j \varphi(b_j)}{z-b_j} - \ldots$$

Consequently, $F(z)$ has a simple pole at the point $z = b_j$, with a residue equal to $-\beta_j\varphi(b_j)$ [which vanishes if $\varphi(b_j) = 0$].

Applying the theorem of residues to the integral

$$\frac{1}{2\pi i}\int_\Gamma \varphi(z)\frac{f'(z)}{f(z)-A}\,dz,$$

we obtain

$$\frac{1}{2\pi i}\int_\Gamma \varphi(z)\frac{f'(z)\,dz}{f(z)-A} = \sum_{j=1}^{m}\alpha_j\varphi(a_j) - \sum_{j=1}^{n}\beta_j\varphi(b_j). \tag{5}$$

The first sum on the right-hand side here represents the sum of the values of the function $\varphi(z)$ at the A-points of the function $f(z)$, where each of the values in the sum is repeated as many times as the multiplicity of the corresponding A-point. If we remember that in the sequence of A-points lying in the interior of Γ, each one of them is repeated as many times as the corresponding multiplicity, then the sum $\sum_{j=1}^{m}\alpha_j\varphi(a_j)$ can be called simply the sum of the values of the function $\varphi(z)$ at the A-points of the function $f(z)$. The same remarks apply also to the second sum, in which the summation includes values at the poles of the function $f(z)$. Finally, we arrive at the following formulation.

The integral $\frac{1}{2\pi i}\int_\Gamma \varphi(z)\frac{f'(z)}{f(z)-A}\,dz$ *is equal to the difference between the sum of the values taken by the function* $\varphi(z)$ *at the A-points of the function* $f(z)$, *lying in the interior of* Γ, *and the sum of the values taken up by the same function* $\varphi(z)$ *at the poles of the function* $f(z)$, *lying again in the interior of* Γ.

We now consider certain special cases of this proposition.

(*a*) $\varphi(z) = z$. In this case, we obtain the formula

$$\frac{1}{2\pi i}\int_\Gamma z\frac{f'(z)}{f(z)-A}\,dz = \sum_{j=1}^{m}\alpha_j a_j - \sum_{j=1}^{n}\beta_j b_j, \tag{6}$$

i.e. the integral is found to be equal to the difference between the sum of the A-points of the function $f(z)$, lying in the interior of Γ, and the sum of the poles of this function, lying again in the interior of Γ.

(*b*) $\varphi(z) \equiv 1$. In this case, we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)-A}\,dz = \sum_{j=1}^{m}\alpha_j - \sum_{j=1}^{n}\beta_j, \tag{7}$$

i.e. the integral is found to be equal to the difference between the number of A-points of the function $f(z)$, lying in the interior of Γ, and the number of its poles, also lying in the interior of Γ.

If $A = 0$, then the A-points will be the zeros of the function $f(z)$. Denoting their number in the interior of Γ by N and the number of poles of the function $f(z)$, lying in the interior of Γ by P, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz = N - P. \tag{8}$$

The integral on the left-hand side is called the *logarithmic residue* of the function $f(z)$ with respect to the contour Γ (we note that the logarithmic derivative of the function $f(z)$ appears under the integral sign). Thus, we arrive at the following theorem :

The difference between the number of zeros and of poles of the function $f(z)$, lying in the interior of the contour Γ (both the zeros and the poles are counted as many times as their multiplicities), is equal to the logarithmic residue of the function with respect to this contour.

The logarithmic residue of the function has a simple meaning. To clarify this, we rewrite the integral in the form

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\Gamma}\frac{d}{dz}\{\operatorname{Ln}[f(z)]\}\,dz.$$

We mark on the curve Γ an arbitrary point z_0, which we shall take to be the initial and the final point of the path of integration. As the curve Γ is traversed by the point z in the positive direction, $\operatorname{Ln} f(z)$ will continuously change and after one complete traversal along the curve its value at the point z_0 will, in general, be different from the original value at this point. But $f(z_0)$ being one and the same, the values of $\operatorname{Ln} f(z_0)$ can differ only due to the different values ascribed to $\operatorname{Arg} f(z_0)$ before and after the traversal. Denoting the initial value of $\operatorname{Arg} f(z_0)$ by Φ_0 and

its value after the traversal by Φ_1, we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\{[\ln|f(z_0)|+i\Phi_1]-[\ln|f(z_0)|+i\Phi_0]\} = \frac{\Phi_1-\Phi_0}{2\pi}.$$

Hence, by formula (8),

$$N-P=\frac{\Phi_1-\Phi_0}{2\pi} \tag{9}$$

or, denoting $\Phi_1-\Phi_0$ by the symbol $\operatorname{Var}_{\Gamma}\operatorname{Arg} f(z)$ (from the Latin word *variatio*, meaning change), we get

$$N-P=\frac{1}{2\pi}\operatorname{Var}_{\Gamma}\operatorname{Arg} f(z).$$

This relation expresses the so-called *principle of the argument*:

The difference between the number of zeros and poles of the function $f(z)$, lying in the interior of the closed curve Γ, is equal to the change in $\operatorname{Arg} f(z)$, while the point z moves once along the contour Γ in the positive direction, divided by 2π.

We further note a geometrical interpretation of this proposition. As the point z moves along the closed curve Γ in the positive direction, the end of the vector $w=f(z)$ describes a certain closed curve Γ'. We denote by ν the number of complete traversals about the origin of coordinates made by the vector w during the traversal in question. A traversal will be counted as $+1$ if it is made in the positive direction and -1 if it is made in the negative direction. Then, for the change in $\operatorname{Arg} f(z)$, we obtain the magnitude $2\pi\nu$, whence follows the following formulation of the principle of the argument:

The difference between the number of zeros and poles of the one-valued function $f(z)$ that lie in the interior of the closed curve Γ is equal to the number of complete traversals ν made about the origin of coordinates by the vector representing $f(z)$, while the point z describes (once) the contour Γ in the positive direction.

In the special case when $f(z)$ has no poles inside Γ we find that *the number of zeros of the function $f(z)$ that lie in the interior of the closed curve Γ is equal to the number of complete traversals of the vector $f(z)$ about the origin of coordinates as the point z moves once along the contour Γ in the positive direction.*

From the principle of the argument, the following theorem is obtained.

The theorem of Rouche'. *If $f(z)$ and $\varphi(z)$ are two functions, one-valued and analytic at the points on a rectifiable closed curve Γ and also*

in its interior and if the condition $|f(z)| > |\varphi(z)|$ *holds at the points of this curve, then, in the interior of* Γ, *the sum* $f(z) + \varphi(z)$ *has the same number of zero as the function* $f(z)$ *has.*

Proof. For finding the number of zeros of the function $f(z) + \varphi(z)$, we shall make use of the principle of the argument. Rewriting $f(z) + \varphi(z)$, for the points of the curve Γ, in the form

$$f(z) + \varphi(z) = f(z)\left[1 + \frac{\varphi(z)}{f(z)}\right]$$

[$|f(z)|$ at the points of curve Γ is greater than $|\varphi(z)|$ and, hence, does not vanish], we obtain

$$\operatorname{Arg}[f(z) + \varphi(z)] = \operatorname{Arg} f(z) + \operatorname{Arg}\left[1 + \frac{\varphi(z)}{f(z)}\right].$$

But $\left|\frac{\varphi(z)}{f(z)}\right| < 1$; hence, the end of the vector representing $1 + \frac{\varphi(z)}{f(z)}$ describes a closed curve, wholly contained in the interior of a circle with its centre at the point 1 and of radius 1. Therefore, the corresponding vector does not revolve around the origin of coordinates, and the change in $\operatorname{Arg}\left[1 + \frac{\varphi(z)}{f(z)}\right]$, as the point z moves along the curve Γ, is equal to zero. Thus, the change in $\operatorname{Arg}[f(z) + \varphi(z)]$ during the traversal mentioned above is the same as the change in $\operatorname{Arg} f(z)$ during the same traversal ; hence, by the principle of the argument, we find that the number of zeros of the functions $f(z) + \varphi(z)$ and $f(z)$ is the same.

The following proposition is a useful application of this theorem.

The theorem of Hurwitz. *If* $\{f_n(z)\}$ *is a sequence of functions, analytic in the domain G, which converges uniformly in the interior of this domain to some function* $f(z) \not\equiv 0$, *then, for any closed rectifiable curve* γ, *lying in G along with its interior and not passing through the zeros of the function* $f(z)$, *we can find such a number* $\nu = \nu(\gamma)$ *that for* $n > \nu(\gamma)$ *each of the functions* $f_n(z)$ *will have in the interior of* γ *exactly the same number of zeros as those of the function* $f(z)$ *that lie in the interior of this curve.*

Proof. We shall denote by μ the minimum value of $|f(z)|$ at the points of the curve γ. By assumption, $\mu > 0$. Hence, because of the uniform convergence of the sequence $\{f_n(z)\}$ on γ, we can find such a $\nu(\gamma)$ that, for $n > \nu(\gamma)$, the inequality

$$|f_n(z) - f(z)| < \mu \leqslant |f(z)|$$

holds at all the points of the curve γ. But it follows from the Rouche' theorem that the functions $f(z)$ and $f(z) + [f_n(z) - f(z)] = f_n(z)$,

$[n > \nu(\gamma)]$, possess an equal number of zeros in the interior of γ. And this proves the theorem.

Example 1. Find the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$, with moduli less than unity.

Here, we use the Rouche' theorem.

For this purpose we express $z^8 - 4z^5 + z^2 - 1$ in the form $f(z) + \varphi(z)$, where $f(z) = -4z^5$ and $\varphi(z) = z^8 + z^2 - 1$. Since for $|z| = 1$, $|\varphi(z)| = |z^8 + z^2 - 1| \leqslant |z^8| + |z^2| + 1 = 3$ and $|f(z)| = |4z^5| = 4$, we have

$$|\varphi(z)| < |f(z)|.$$

Hence, by the Rouche' theorem, the function $f(z) + \varphi(z) = z^8 - 4z^5 + z^2 - 1$ has, in the interior of the circle $|z| = 1$, the same number of zeros as the function $f(z) = -4z^5$. But this latter function has a zero with multiplicity five at the origin of coordinates; hence, the number of its zeros in the unit circle is equal to five. Therefore, the equation $z^8 - 4z^5 + z^2 - 1 = 0$ also has five roots in the interior of the unit circle, i.e. five roots with moduli less than unity.

Example 2. Prove that the equation

$$a_0 + a_1 \cos \vartheta + a_2 \cos 2\vartheta + \ldots + a_n \cos n\vartheta = 0,$$

where $0 < a_0 < a_1 < \ldots < a_n$, has, in the interval $0 < \vartheta < 2\pi$, $2n$ different roots. Moreover, show that the given equation does not possess any imaginary root.

We first prove that all the zeros of the polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

lie in the interior of the unit circle. This polynomial, evidently, does not have any real positive roots. If z is not a positive number, then

$$\begin{aligned} |p(z)(z-1)| &= |a_n z^{n+1} - [a_0 + (a_1 - a_0) z + \ldots + (a_n - a_{n-1}) z^n]| \geqslant \\ &\geqslant |a_n z^{n+1}| - |a_0 + (a_1 - a_0) z + \ldots + (a_n - a_{n-1}) z^n| > \\ &> a_n |z|^{n+1} - [a_0 + (a_1 - a_0)|z| + \ldots + (a_n - a_{n-1})|z|^n]. \end{aligned}$$

In fact, since the numbers a_0, $a_1 - a_0$, ..., $a_n - a_{n-1}$ are positive and the number z is not positive, the vectors a_0, $(a_1 - a_0) z$, ..., $(a_n - a_{n-1}) z^n$ cannot be directed in one and the same sense; consequently,

$$\begin{aligned} |a_0 + (a_1 - a_0) z + \ldots + (a_n - a_{n-1}) z^n| &< a_0 + (a_1 - a_0)|z| + \ldots \\ &\ldots + (a_n - a_{n-1})|z|^n. \end{aligned}$$

If, however, $|z| \geqslant 1$, then

$$a_0 + (a_1-a_0)|z| + \ldots + (a_n-a_{n-1})|z|^n \leqslant$$
$$\leqslant a_0|z|^{n+1} + (a_1-a_0)|z|^{n+1} + \ldots + (a_n-a_{n-1})|z|^{n+1} =$$
$$= [a_0+(a_1-a_0) + \ldots + (a_n-a_{n-1})]|z|^{n+1} = a_n|z|^{n+1}.$$

Thus, when $|z| \geqslant 1$ and z is nonpositive,

$$|p(z)(z-1)| > a_n|z|^{n+1} - a_n|z|^{n+1} = 0, \text{ i.e. } p(z)(z-1) \neq 0.$$

But from this it follows that when z is nonpositive and its modulus is not less than unity, $p(z) \neq 0$. The latter holds also for positive values of z, and, consequently, $p(z)$ does not have zeros either in the exterior of the unit circle or on it. Hence, all the n zeros of the polynomial $p(z)$ lie strictly within the interior of the unit circle.

We make the point z describe the circle $|z| = 1$ in the positive direction. Then, the vector representing $p(z)$ must, by the principle of the argument, perform about the origin of coordinates such a number of traversal as is equal to the number of zeros of the polynomial $p(z)$, i.e. n. Since in every traversal the curve described by the end of the vector intersects the imaginary axis at least twice (once from above and once from below), we have at least $2n$ such intersections. Each of these will correspond to a definite position of the point z on the circle $|z| = 1$, i.e. to a definite value of its argument ϑ which changes during one revolution over the interval $(0, 2\pi)$.

Hence, we have at least $2n$ different values of the argument ϑ in the interval $0 < \vartheta < 2\pi$, for which the point representing $p(z) = p(e^{i\vartheta})$ falls on the imaginary axis. For each of these values of ϑ,

$$\operatorname{Re}[p(e^{i\vartheta})] = \operatorname{Re}(a_0+a_1e^{i\vartheta} + \ldots + a_ne^{in\vartheta}) =$$
$$= \operatorname{Re}[a_0+a_1(\cos\vartheta+i\sin\vartheta) + \ldots + a_n(\cos n\vartheta+i\sin n\vartheta)] =$$
$$= a_0 + a_1\cos\vartheta + \ldots + a_n\cos n\vartheta,$$

reduces to zero. This proves the existence of at least $2n$ roots of the equation

$$a_0 + a_1\cos\vartheta + \ldots + a_n\cos n\vartheta = 0$$

in the interval $(0, 2\pi)$.

We shall now show that the number of roots in this interval is exactly equal to $2n$. To prove this, we put $e^{i\vartheta} = \zeta$. Thus, we have

$$\cos k\vartheta = \frac{e^{ik\vartheta}+e^{-ik\vartheta}}{2} = \frac{\zeta^k+\zeta^{-k}}{2},$$

and, consequently,

$$a_0 + a_1 \cos \vartheta + \ldots + a_n \cos n\vartheta = \frac{1}{2} \zeta^{-n} (a_n + a_{n-1}\zeta + \ldots + a_1\zeta^{n-1} + 2a_0\zeta^n + a_1\zeta^{n+1} + \ldots + a_n\zeta^{2n}).$$

If $\zeta_1, \zeta_2, \ldots, \zeta_{2n}$ are the zeros of the polynomial on the right-hand side, then all the zeros of the trigonometric polynomial on the left will satisfy the relations

$$e^{i\vartheta} = \zeta_j \qquad (j=1, 2. \ldots, 2n).$$

From this, it immediately follows that the number of different real zeros of the given trigonometric polynomial in the interval $(0, 2\pi)$ cannot exceed $2n$. Since we proved above the existence of at least $2n$ different real zeros of this polynomial in the interval $(0, 2\pi)$, their total number must be exactly equal to $2n$. We also find that the moduli of the numbers $\zeta_j = e^{i\vartheta_j}$ are all equal to unity. Hence, there cannot be any imaginary zero among the various zeros of the given trigonometric polynomial.

3. RESIDUE AT THE POINT AT INFINITY

If $f(z)$ is one-valued and analytic in some neighbourhood $|z| > R$ of the point at infinity (excluding at most this point itself), then its expansion

$$f(z) = \ldots + A_{-m}z^{-m} + \ldots + A_{-1}z^{-1} + A_0 + A_1z + \ldots + A_nz^n + \ldots$$

will be valid in this neighbourhood.

We shall now integrate $f(z)$ along the circle C_σ: $|z| = \sigma$, where $\sigma > R$, the direction of traversal being so chosen that the neighbourhood $|z| > \sigma$ of the point at infinity would lie, during the traversal, to the left. Such a direction will naturally be taken as positive with respect to traversal around the point at infinity, but with respect to the interior of this circle : $|z| < \sigma$, i.e. with respect to the neighbourhood of the finite point 0, it will be negative. By a termwise integration of the Laurent series, we get

$$\int_{C_\sigma} f(z)\, dz = A_{-1}(-2\pi i) = 2\pi i\,(-A_{-1}).$$

In order that in this case of integration round the point at infinity the integral of the function be equal to $2\pi i$ times the residue of the function, it will be expedient to introduce the following definition : *The residue*

of a function, which is one-valued and analytic in some neighbourhood of the point $z = \infty$, at this point, is defined to be the coefficient of z^{-1} (with the negative sign) in the Laurent expansion of the function in this neighbourhood.

Then

$$\int_{C_\sigma} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z),$$

where the integral is taken in a direction positive with respect to the point at infinity, i.e. in a direction for which not the interior of the curve but its exterior will lie to the left.

Making use of this definition, we obtain the following theorem.

The sum of all the residues of a one-valued analytic function, possessing only isolated singularities in the extended plane, is equal to zero.

It immediately follows that the number of singularities of such a function is finite (otherwise, there would exist a finite limit point, or a limit point at infinity, of the set of singularities, and this point would be a nonisolated singularity of the function). We describe a circle $|z| = \sigma$ with its centre at the origin of coordinates, such that there would be no singularities of the function either on it or in its exterior (excluding at most the point $z = \infty$). Then, all the finite singular points $z_1, z_2, \ldots, z_n$ will lie in the interior of this circle ; thus, by the theorem of residues, we shall have

$$\int_{C_\sigma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Here, the integral is taken in the usual positive direction, i.e. in a direction with respect to which the interior of the circle remains to the left. But the same direction will be negative with respect to the point at infinity. Hence, the same integral will be equal to

$$\int_{C_\sigma} f(z)\,dz = +\,2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

Subtracting the first equation from the second, we finally get

$$2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \ldots + \operatorname*{Res}_{z=z_n} (z) + \operatorname*{Res}_{z=\infty} f(z)\right] = 0,$$

or

$$\operatorname*{Res}_{z=z_1} f(z) + \ldots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

Thus, the theorem is proved.

This theorem is true, in particular, for any rational function because a rational function has only isolated singularities of a simple character (that is, poles).

We find that the residue of a function at the point at infinity is determined by the coefficient of one of the terms of the *regular part* of the Laurent expansion, while the residue at a finite point is determined by the coefficient of one of the terms of the *principal part* (we should remember that the totality of the negative powers of the Laurent series represents the regular part for the point $z = \infty$ but the principal part for a finite point). Hence, it follows that the residue at the point $z = \infty$ can be nonzero even when this point is not a singularity, i.e. when this point is a regular point, whereas the residue at a finite regular point is always equal to zero. Thus, for instance, the point $z = \infty$ is a regular point (zero of the first order) of the function $f(z) = 1/z$. At the same time, in this case, $\operatorname{Res}_{z=\infty} f(z) = -1 \neq 0$.

4. APPLICATION OF THE THEOREM OF RESIDUES TO THE EXPANSION OF MEROMORPHIC FUNCTIONS INTO PARTIAL FRACTIONS

Let $f(z)$ be a one-valued analytic function not possessing, in a finite part of the plane, singularities other than poles.

We denote by C any closed rectifiable Jordan curve not passing through the poles of the function $f(z)$, and let z be a point in the interior of C, being different from the origin of coordinates and the poles. We now calculate the integral

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z};$$

this is similar in form to the Cauchy integral. But this is not exactly the Cauchy integral because the function $f(z)$, being analytic on the contour C, has singularities (poles) in the interior of C. This integral, at any rate, belongs to the class of the Cauchy type integrals. In general, an integral of the form

$$\frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)\,d\zeta}{\zeta - z}$$

is referred to as a *Cauchy type integral*, the function $\varphi(\zeta)$ being continuous on the rectifiable curve L. In the special case, when L is a closed rectifiable

Jordan curve belonging, along with its interior $I(L)$, to some domain in which $\varphi(z)$ is a one-valued analytic function, a Cauchy type integral becomes a customary Cauchy integral. We now turn to the integral under discussion.

Obviously, the point $\zeta = z$ and all the poles of the function $f(z)$ that lie in the interior of C are the poles of the function $\varphi(\zeta) = \frac{f(\zeta)}{\zeta - z}$ in the interior of C. We denote those poles which are different from zero by $\beta_1, \ldots, \beta_n$, and the principal parts of the Laurent expansions of the function $f(z)$ about them by $G_1(z), \ldots, G_n(z)$. We further put $\beta_0 = 0$, setting $G_0(z)$ equal to the principal part of the Laurent expansion of the function $f(z)$ in the neighbourhood of the point $z = 0$, so that the function $G_0(z)$ is identically equal to zero in case $z = 0$ is a regular point of $f(z)$, whereas $G_0(z)$ is a rational function, with a unique pole at the origin, in case $z = 0$ is a pole of the function $f(z)$.

We now calculate the residues of the function $\varphi(\zeta)$ at the points $\zeta = z, \beta_0, \ldots, \beta_n$. To begin with, we obtain

$$\operatorname*{Res}_{\zeta=z} \varphi(\zeta) = f(z).$$

Further, replacing $f(\zeta)$ in the neighbourhood of the point β_k by the corresponding Laurent expansion

$$f(\zeta) = \frac{A_{-\gamma_k}^{(k)}}{(\zeta-\beta_k)^{\gamma_k}} + \ldots + \frac{A_{-1}^{(k)}}{\zeta-\beta_k} + \\ + A_0^{(k)} + A_1^{(k)}(\zeta-\beta_k) + \ldots = G_k(\zeta) + P_k(\zeta),$$

where $G_k(\zeta)$ and $P_k(\zeta)$ are the principal and regular parts, respectively, and observing that

$$\frac{1}{\zeta-z} = -\frac{1}{(z-\beta_k)-(\zeta-\beta_k)} = \\ = -\frac{1}{z-\beta_k} - \frac{\zeta-\beta_k}{(z-\beta_k)^2} - \ldots - \frac{(\zeta-\beta_k)^{\gamma_k-1}}{(z-\beta_k)^{\gamma_k}} - \ldots,$$

for $|\zeta-\beta_k| < |z-\beta_k|$, we find that the term containing $(\zeta-\beta_k)^{-1}$ in the expansion $\varphi(\zeta) = f(\zeta)\frac{1}{\zeta-z}$ is equal to

$$-\left[\frac{A_{-1}^{(k)}}{z-\beta_k} + \frac{A_{-2}^{(k)}}{(z-\beta_k)^2} + \ldots + \frac{A_{-\gamma_k}^{(k)}}{(z-\beta_k)^{\gamma_k}}\right]\frac{1}{\zeta-\beta_k} = -G_k(z)\frac{1}{\zeta-\beta_k}.$$

Consequently,

$$\operatorname*{Res}_{\zeta=\beta_k} \varphi(\zeta) = -G_k(z).$$

Thus,

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} = f(z) - \sum_0^n G_k(z)$$

or

$$f(z) = \sum_0^n G_k(z) + \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{\zeta - z}. \tag{10}$$

This formula might also be derived in a different way. We note, for example, that $\sum_0^n G_k(\zeta)$ is a rational function which vanishes at infinity and that all its poles lie in the interior of C. The function $\frac{1}{\zeta - z}\sum_0^n G_k(\zeta)$ (z lies in the interior of C) possesses the same properties and has at infinity a zero which is at least of the second order. Hence, its residue at the point at infinity will be equal to zero and, consequently,

$$\frac{1}{2\pi i}\int_C \frac{1}{\zeta - z}\sum_0^n G_k(\zeta)\,d\zeta = 0,$$

whence

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_C \frac{1}{\zeta - z}\left[f(\zeta) - \sum_0^n G_k(\zeta)\right]d\zeta.$$

But the function $\left[f(\zeta) - \sum_0^n G_k(\zeta)\right]$ is analytic at all points in the interior of C; hence the Cauchy formula is applicable to the last integral. We thus obtain

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_C \frac{1}{\zeta - z}\left[f(\zeta) - \sum_0^n G_k(\zeta)\right]d\zeta = f(z) - \sum_0^n G_k(z),$$

i.e. the formula (10).

We now assume that there exists a sequence $\{C_m\}$ of closed rectifiable Jordan curves not passing through the poles of the function $f(z)$, such that each (C_m) lies within the next (C_{m+1}) and that the interiors of these

curves, for sufficiently large values of m, contain any given circle $|z| < R$, where the condition

$$\lim_{m\to\infty} \frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0 \tag{11}$$

is satisfied for the curves C_m.

Then, the number of poles of the function $f(z)$, lying in the interior of C_m, will depend on $m : n = n_m$, and, from the formula (10), we obtain

$$f(z) = \lim_{m\to\infty} \sum_{0}^{n_m} G_k(z), \tag{12}$$

i.e. the function $f(z)$ can be represented as the limit of a sequence of the sums of the principal parts of its Laurent expansions at the poles lying in the interior of C_m.

For instance, the condition (11) is satisfied when

$$\overline{\lim_{m\to\infty}} \int_{C_m} |f(\zeta)|\,ds < \infty. \tag{13}$$

Indeed, denoting by r_m the distance from the origin of coordinates to C_m ($r_m \to \infty$ as $m \to \infty$) and assuming that z belongs to the circle $|z| < R$, we obtain, for $r_m > R$,

$$\left|\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)\,d\zeta}{\zeta - z}\right| < \frac{1}{2\pi(r_m - R)}\int_{C_m} |f(\zeta)|\,ds \to 0 \quad \text{as } m \to \infty.$$

From this it can be seen that, under the condition (13), the remainder term of the formula (10) tends to zero uniformly with respect to z if it belongs to an arbitrary circle $|z| < R$. Hence, the sequence (12) converges uniformly to the function $f(z)$ in any circle $|z| < R$.

We can obtain for $f(z)$ an expression analogous to (12) under far more general conditions. For instance, let us assume that for the sequence of curves $\{C_m\}$ and for a certain integral nonnegative number p the relation

$$\overline{\lim_{m\to\infty}} \int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\,ds < \infty \tag{14}$$

holds in place of (13).

If we assume that the lengths l_m of the curves C_m do not increase faster than λr_m, where λ is some constant (it will be so in all cases when C_m are curves similar with respect to the origin of coordinates), then we can derive

$$\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\,ds < \max_{C_m}|f(\zeta)|\,\frac{l_m}{r_m^{p+1}} < \lambda\,\frac{\max\limits_{C_m}|f(\zeta)|}{r_m^p}.$$

Hence, it is clear that the condition (14) will be satisfied if the much simpler condition

$$\overline{\lim_{m\to\infty}}\,\frac{\max\limits_{C_m}|f(\zeta)|}{r_m^p} < \infty, \tag{15}$$

is satisfied. This condition implies an indefinite increase in $\max\limits_{C_m}|f(\zeta)|$, but not faster than that of r_m^p.

Assuming that the condition (14) [or (15)] is satisfied, we turn to the relation (10) and replace in it the fraction involving $\zeta - z$ under the integral sign by the expression

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\,\frac{1}{1-\frac{z}{\zeta}} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \ldots + \frac{z^p}{\zeta^{p+1}} + \frac{1}{(\zeta - z)}\,\frac{z^{p+1}}{\zeta^{p+1}}.$$

In this case, we shall have

$$f(z) = \sum_0^n G_k(z) = \sum_0^p \frac{z^j}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta + \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\frac{z^{p+1}}{\zeta^{p+1}}\,d\zeta. \tag{16}$$

Noting that the function $\frac{f(\zeta)}{\zeta^{j+1}}$ has the points $\beta_0, \beta_1, \ldots, \beta_n$ as its poles in the interior of C, we can put

$$\operatorname*{Res}_{z=\beta_k}\frac{f(\zeta)}{\zeta^{j+1}} = A_k^{(j)} \quad (j=0,\ 1, \ldots, p).$$

Then, we obtain

$$\sum_{j=0}^p \frac{z^j}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta = \sum_{j=0}^p \left(A_0^{(j)} + \ldots + A_n^{(j)}\right) z^j = \sum_{k=0}^n P_k(z),$$

where $P_k(z)$ are polynomials of degree not higher than p:

$$P_k(z) = A_k^{(0)} + A_k^{(1)} z + \ldots + A_k^{(p)} z^p. \tag{17}$$

Thus, the formula (16) can be rewritten in the form

$$f(z) = \sum_{0}^{n}\Big[G_k(z) + P_k(z) \Big] + \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta. \tag{18}$$

Replacing C by C_m and, hence, n by n_m, we find by using the condition (14) that the remainder term of this formula,

$$\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)}{\zeta - z}\frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta,$$

tends to zero, and besides uniformly at all points z that belong to any fixed circle $|z| < R$. In fact, when m is so large that $r_m > R$, we have

$$\left|\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)}{\zeta - z}\frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta\right| < \frac{1}{2\pi}\int_{C_m} \frac{|f(\zeta)|}{|\zeta| - |z|}\left|\frac{z}{\zeta}\right|^{p+1} ds < \frac{R^{p+1}}{2\pi (r_m - R)}\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds.$$

But, by the condition (14), the integrals $\int_{C_m} \frac{|f(z)|}{|\zeta|^{p+1}}\, ds$ are bounded :

$$\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds < M < \infty.$$

Hence

$$\left|\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)}{\zeta - z}\frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta\right| < \frac{MR^{p+1}}{2\pi (r_m - R)} \to 0 \quad \text{as} \quad M \to \infty .$$

We also obtain from the formula (18) the expansion

$$f(z) = \lim_{n_m \to \infty} \sum_{0}^{n_m} [G_k(z) + P_k(z)], \tag{19}$$

which converges uniformly to $f(z)$ in any circle $|z| < R$. This expansion may also be written in the form of the series

$$f(z) = [G_0(z) + P_0(z)] + \sum_{m=0}^{\infty} \{[G_{n_m+1}(z) + P_{n_m+1}(z)] + \ldots$$
$$\ldots + [G_{n_{m+1}}(z) + P_{n_{m+1}}(z)]\}, \tag{20}$$

where n_0 has to be replaced by zero.

We observe that the first term of the sequence (19) or of the series (20) becomes ∞ at the points $\beta_0, \ldots, \beta_{n_m}$, i.e. where the function $f(z)$ becomes ∞. Hence, the uniform convergence of the series (19) can be looked upon as the uniform convergence of that series which is obtained therefrom by omitting the various first terms that have poles in the circle $|z| < R$.

Expansions having the form (19) [especially (12) and (20)] are expansions of $f(z)$ into partial fractions.

5. EXPANSION OF sec z, cot z, cosec z, AND tan z INTO PARTIAL FRACTIONS

The foregoing method of expansion of functions into series owes its origin to Cauchy. We shall apply it to several special cases of considerable importance.

(*a*) **The expansion of** sec z. For the C_m we choose square contours with their centres at the point $z = 0$ and sides parallel to the axes of coordinates and equal in length to $2m\pi$. We have : $z = \pm m\pi + iy$ on the sides of the squares which are parallel to the imaginary axis, and hence

$$|\sec z| = \frac{1}{|\cos(\pm m\pi + iy)|} = \frac{1}{|\cos iy|} = \frac{1}{\cosh y}.$$

On the sides of the squares which are parallel to the real axis, we have : $z = x \pm im\pi$ and, consequently, [vide formula (36) of Chapter 3],

$$|\sec z| = \frac{1}{|\cos(x \pm im\pi)|} \leqslant \frac{1}{\sinh m\pi}.$$

From these inequalities we obtain for the integral $\int\limits_{C_m} |\sec \zeta|\, ds$ the estimate

$$\int\limits_{C_m} |\sec \zeta|\, ds < 2 \int\limits_{-m\pi}^{m\pi} \frac{dy}{\cosh y} + 4m\pi \frac{1}{\sinh m\pi}.$$

Since the integral $\int\limits_{-\infty}^{+\infty} \frac{dy}{\cosh y}$ converges and $\frac{4m\pi}{\sinh m\pi} \to 0$ as $m \to \infty$, the condition (13) and, hence, also the condition (11) are satisfied. So, in the given case, we can use the formula (12).

The function $\sec z = \frac{1}{\cos z}$ in the interior of the C_m has poles given

by $(2j-1)\frac{\pi}{2}$, where $-m+1 \leqslant j \leqslant m$; all these poles are simple, because the zeros of cos z are all simple. Evidently

$$\operatorname*{Res}_{z=(2j-1)\frac{\pi}{2}} \sec z = -\frac{1}{\sin\left(\frac{2j-1}{2}\pi\right)} = (-1)^j,$$

and, consequently, the principal part of sec z, in the neighbourhood of the point $z = (2j-1)\frac{\pi}{2}$, is $G_j(z) = \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}}$. We also note that $z = 0$ is not a pole of sec z and, consequently, the corresponding principal part has to be taken as equal to zero. We find from the formula (12) that

$$\sec z = \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} =$$

$$= \lim_{m\to\infty} \left\{ \sum_{j=1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} + \sum_{j=-m+1}^{0} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} \right\}.$$

We replace j by $1-k$ in the second of the sums under the limit sign; thereby we find that k varies between the limits 1 and m. Hence,

$$\sum_{-m+1}^{0} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{z+(2k-1)\frac{\pi}{2}}.$$

Thus, replacing the symbol k by j, we obtain

$$\sec z = \lim_{m\to\infty} \left\{ \sum_{j=1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} + \sum_{j=1}^{m} \frac{(-1)^{j+1}}{z+(2j-1)\frac{\pi}{2}} \right\} =$$

$$= \lim_{m\to\infty} \sum_{j=1}^{m} (-1)^j \frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}.$$

Finally, we arrive at the expansion of sec z into the series

$$\sec z = \sum_{j=1}^{\infty} (-1)^j \frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}. \tag{21}$$

From the method of derivation of this series [a special case of the formula (12) under the condition (13)] it follows that it converges uniformly in any circle $|z| < R$ (and, in this case, in order that we can speak of its convergence, we have to exclude from it some of the first terms having poles in the given circle).

(*b*) **The expansion of** $\cot z$. For the C_m we choose the square contours with their centres at the point $z = 0$ and with sides parallel to the coordinate axes and equal in length to $(2m+1)\frac{\pi}{2}$. Then, on the sides of the squares that are parrallel to the imaginary axis, we have

$$z = \pm\left(m+\frac{1}{2}\right)\pi + iy,$$

whereby

$$|\cot z| = \left|\frac{\cos\left[\pm\left(m+\frac{1}{2}\right)\pi+iy\right]}{\sin\left[\pm\left(m+\frac{1}{2}\right)\pi+iy\right]}\right| = \left|\frac{\sin(iy)}{\cos(iy)}\right| = \left|\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right| < 1.$$

On the sides of squares that are parallel to the real axis, we have

$$z = x \pm i\left(m+\frac{1}{2}\right)\pi,$$

and, consequently [vide formula (36) of Chapter 3],

$$|\cot z| = \left|\frac{\cos\left[x\pm i\left(m+\frac{1}{2}\right)\pi\right]}{\sin\left[x\pm i\left(m+\frac{1}{2}\right)\pi\right]}\right| \leqslant \frac{\cosh\left(m+\frac{1}{2}\right)\pi}{\sinh\left(m+\frac{1}{2}\right)\pi} =$$

$$= \frac{1+e^{-(2m+1)\pi}}{1-e^{-(2m+1)\pi}} \leqslant \frac{1+e^{-\pi}}{1-e^{-\pi}} = \frac{e^{\pi}+1}{e^{\pi}-1}.$$

Hence, on the sides of the squares C_m the modulus $|\cot z|$ satisfies the inequality

$$|\cot z| \leqslant \frac{e^{\pi}+1}{e^{\pi}-1}.$$

Hence, the condition (15) and, hence, also the condition (14) are satisfied for $p = 0$, and we can apply formula (19).

The function $\cot z = \frac{\cos z}{\sin z}$ has poles at $0, \pm\pi, \ldots, \pm m\pi$, in the

interior of the C_m ; in this case, these poles will be all simple because all the zeros of sin z are so. Evidently,

$$\operatorname*{Res}_{z=k\pi} \cot z = \frac{\cos k\pi}{\cos k\pi} = 1 ;$$

hence, the principal part $G_k(z)$ of the expansion of cot z in the neighbourhood of the point $z = k\pi$ is equal to $\frac{1}{z-k\pi}$.

The polynomials $P_k(z)$ [vide formula (17)] are, in the given case, polynomials of degree not higher than $p = 0$:

$$P_k(z) = A_k^{(0)} = \operatorname*{Res}_{\zeta=k\pi} \frac{\cot \zeta}{\zeta}.$$

However, the function $\frac{\cot \zeta}{\zeta}$ is, obviously, even. Hence, its expansion into a Laurent series in the neighbourhood of the origin of coordinates contains only even powers of ζ ; consequently,

$$\operatorname*{Res}_{\zeta=0} \frac{\cot \zeta}{\zeta} = 0.$$

Moreover, the points $\zeta = k\pi$ $(k \neq 0)$ are simple poles for $\frac{\cot \zeta}{\zeta}$.

Hence

$$\operatorname*{Res}_{\zeta=k\pi} \frac{\cot \zeta}{\zeta} = \frac{\cos k\pi}{k\pi \cos k\pi} = \frac{1}{k\pi}.$$

Thus

$$P_0(z) = 0, \quad P_k(z) = \frac{1}{k\pi} \ (k \neq 0) ;$$

consequently, by formula (19),

$$\cot z = \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{k\pi} \right) + \sum_{k=1}^{m} \left(\frac{1}{z+k\pi} - \frac{1}{k\pi} \right) \right] =$$

$$= \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right) \right] = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right). \tag{22}$$

We have thus obtained an expansion of cot z into partial fractions. As it follows from the very method of deriving this formula [from the

general formula (19)], the series (22) converges uniformly in any circle $|z| < R$, provided that we exclude from this series a finite number of terms having poles in this circle.

Rewriting the relation (22) in the form

$$\cot z - \frac{1}{z} = \sum_1^\infty \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi}\right),$$

we integrate termwise along any arbitrary curve L starting from the origin of coordinates and not passing through the points $k\pi$ ($k = \pm 1, \pm 2, \ldots$). We thus obtain

$$\int_0^z \left(\cot z - \frac{1}{z}\right) dz = \sum_1^\infty \operatorname{Ln}\left(\frac{k\pi - z}{k\pi}\,\frac{k\pi + z}{k\pi}\right) = \sum_1^\infty \operatorname{Ln}\left(1 - \frac{z^2}{k^2\pi^2}\right),$$

where the right-hand side contains the fully defined values of the logarithms, i.e. values representing the magnitudes of the corresponding integrals

$$\int_L \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi}\right) dz.$$

The integral on the left-hand side is equal to one of the values of $\operatorname{Ln}\frac{\sin z}{z}$. Thus,

$$\operatorname{Ln}\frac{\sin z}{z} = \lim_{n\to\infty} \sum_1^\infty \operatorname{Ln}\left(1 - \frac{z^2}{k^2\pi^2}\right),$$

whence

$$\frac{\sin z}{z} = \lim_{n\to\infty} \prod_1^\infty \left(1 - \frac{z^2}{k^2\pi^2}\right),$$

which may be written, in terms of the symbol denoting the infinite product,

$$\sin z = z \prod_1^\infty \left(1 - \frac{z^2}{k^2\pi^2}\right). \tag{23}$$

This is the expansion of sin z into an infinite product.

(*c*) **The expansions of** cosec z **and** tan z. From the expansions of sec z and cot z we immediately obtain expansions of cosec z and tan z. In fact, cosec $z = \sec\left(\frac{\pi}{2} - z\right)$. Now, from the formula obtained above,

$$\sec z = \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^j}{z - (2j-1)\frac{\pi}{2}}$$

hence, we obtain

$$\operatorname{cosec} z = \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^j}{\frac{\pi}{2} - z - (2j-1)\frac{\pi}{2}} = \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^j}{-\pi(j-1)-z} =$$

$$= \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^{j-1}}{\pi(-1)+z}.$$

Replacing $j-1$ by k, we have

$$\operatorname{cosec} z = \lim_{m\to\infty} \sum_{k=-m}^{m-1} \frac{(-1)^k}{z+k\pi} = \lim_{m\to\infty} \left[\frac{1}{z} - \left(\frac{1}{z-\pi} + \frac{1}{z+\pi}\right) + \right.$$

$$+ \left(\frac{1}{z-2\pi} + \frac{1}{z+2\pi}\right) - \ldots$$

$$\left. \ldots + (-1)^{m-1}\left(\frac{1}{z-(m-1)\pi} + \frac{1}{z+(m-1)\pi}\right) + (-1)^m \frac{1}{z-m\pi}\right].$$

We add within the square brackets one more term: $(-1)^m \frac{1}{z+m\pi}$, the limit of which is equal to zero (uniform for all z belonging to any circle $|z| < R$). Thus, we obtain

$$\operatorname{cosec} z = \lim_{m\to\infty} \left[\frac{1}{z} - \frac{2z}{z^2-\pi^2} + \frac{2z}{z^2-(2\pi)^2} - \ldots + (-1)^m \frac{2z}{z^2-(m\pi)^2}\right] =$$

$$= \frac{1}{z} + \sum_{1}^{\infty} (-1)^k \frac{2z}{z^2-(k\pi)^2},$$

which is the required result.

Similarly, we have for tan z: $\tan z = \cot\left(\frac{\pi}{2} - z\right)$. Now, from the formula obtained above,

$$\cot z = \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi}\right)\right];$$

hence, we obtain

$$\tan z = \lim_{m\to\infty}\left[\frac{1}{\frac{\pi}{2}-z}+\sum_{k=1}^{m}\left(\frac{1}{\frac{\pi}{2}-z-k\pi}+\frac{1}{\frac{\pi}{2}-z+k\pi}\right)\right]=$$

$$=\lim_{m\to\infty}\left[\left(\frac{1}{\frac{\pi}{2}-z}+\frac{1}{-\frac{\pi}{2}-z}\right)+\left(\frac{1}{\frac{3\pi}{2}-z}+\frac{1}{-\frac{3\pi}{2}-z}\right)+\ldots+\right.$$

$$\left.\ldots+\left(\frac{1}{(2m-1)\frac{\pi}{2}-z}+\frac{1}{-(2m-1)\frac{\pi}{2}-z}\right)+\frac{1}{(2m+1)\frac{\pi}{2}-z}\right].$$

Omitting the term $\dfrac{1}{(2m+1)\frac{\pi}{2}-z}$ within the square brackets, the limit of which is equal to zero, we get

$$\tan z = \lim_{m\to\infty}\sum_{1}^{m}\frac{2z}{z^2-\frac{(2k-1)^2}{4}\pi^2}=\sum_{1}^{\infty}\frac{2z}{z^2-\frac{(2k-1)^2}{4}\pi^2},$$

which is the desired expansion for $\tan z$.

(*d*) From the expansions obtained above, it is easy to derive, in particular, the Laurent expansions of trigonometric functions in a neighbourhood of the origin of coordinates—expansions which were earlier obtained by means of the division of series (Art. 11 of Chapter 6).

We now examine the function $\sec z$. This function is analytic in the circle $|z|<\frac{\pi}{2}$ and, consequently, admits of an expansion into a Taylor series in this circle. The formula (21) represents this function as the sum of a series:

$$\sec z=\sum_{1}^{\infty}(-1)^j\frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}=$$

$$=\sum_{1}^{\infty}(-1)^j\left[\frac{1}{z-(2j-1)\frac{\pi}{2}}-\frac{1}{z+(2j-1)\frac{\pi}{2}}\right],$$

which converges uniformly in any circle—particularly, into the interior of the circle $|z|<\frac{\pi}{2}$. Hence, by the theorem of Weierstrass on the uniformly convergent series of analytic functions, the Taylor coefficients of $\sec z$ can be obtained by adding up the corresponding coefficients of the

Taylor expansions of each of the functions appearing in the square brackets on the right-hand side of the last formula (*cf.* Art. 8, Chapter 6). Now

$$(-1)^j\left[\frac{1}{z-(2j-1)\frac{\pi}{2}}-\frac{1}{z+(2j-1)\frac{\pi}{2}}\right]=$$

$$=-(-1)^j\left[\sum_0^\infty\frac{z^k}{\left[(2j-1)\frac{\pi}{2}\right]^{k+1}}+\sum_0^\infty\frac{(-1)^k z^k}{\left[(2j-1)\frac{\pi}{2}\right]^{k+1}}\right]=$$

$$=2(-1)^{j-1}\sum_0^\infty\frac{z^{2m}}{\left[(2j-1)\frac{\pi}{2}\right]^{2m+1}}\qquad\left(|z|<(2j-1)\frac{\pi}{2}\right);$$

here, the coefficients of odd powers are equal to zero, while those of z^{2m} $(m=0, 1, 2, \ldots)$ are given by

$$\frac{2(-1)^{j-1}}{\left[(2j-1)\frac{\pi}{2}\right]^{2m+1}}=2\left(\frac{2}{\pi}\right)^{2m+1}\frac{(-1)^{j-1}}{(2j-1)^{2m+1}}.$$

Hence, the coefficients of odd powers of z in the Taylor expansion of sec z are also equal to zero (this is quite obvious, because sec z is an even function), while the coefficients of even powers, z^{2m}, can be written in the form of the series

$$2\left(\frac{2}{\pi}\right)^{2m+1}\sum_1^\infty\frac{(-1)^{j-1}}{(2j-1)^{2m+1}}.$$

Thus

$$\sec z=\sum_{m=0}^\infty\left[2\left(\frac{2}{\pi}\right)^{2m+1}\sum_{j=1}^\infty\frac{(-1)^{j-1}}{(2j-1)^{2m+1}}\right]z^{2m}.$$

Let us recall that in Art. 11 of Chapter 6 the same expansion was obtained in a different form, viz.

$$\sec z=\sum_0^\infty(-1)^m\frac{E_{2m}}{(2m)!}z^{2m},$$

where E_{2m} were integers called the *Euler numbers* ($E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61, \ldots$). Comparing the coefficients of the two series, one obtains the equalities

$$2\left(\frac{2}{\pi}\right)^{2m+1}\sum_1^\infty\frac{(-1)^{j-1}}{(2j-1)^{2m+1}}=(-1)^m\frac{E_{2m}}{(2m)!}\qquad(m=0, 1, 2, \ldots),$$

and, in particular,

$$\sum_1^\infty \frac{(-1)^{j-1}}{2j-1} = \frac{E_0}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

$$\sum_1^\infty \frac{(-1)^{j-1}}{(2j-1)^3} = -\frac{E_2}{2\cdot 2!}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{32}.$$

$$\sum_1^\infty \frac{(-1)^{j-1}}{(2j-1)^5} = \frac{E_4}{2\cdot 4!}\left(\frac{\pi}{2}\right)^5 = \frac{5\pi^5}{1536},$$

. . . . . .

We also examine the function $\cot z - \frac{1}{z}$, which is analytic in the circle $|z| < \pi$. In order to evaluate the coefficients in its Taylor expansion, we use the formula (22). From this formula we derive the following series expansion for the given function:

$$\cot z - \frac{1}{z} = \sum_{j=1}^\infty \left(\frac{1}{z-j\pi} + \frac{1}{z+j\pi}\right).$$

For each summand of this sum we have the Taylor expansion

$$\frac{1}{z-j\pi} + \frac{1}{z+j\pi} = -\sum_{k=0}^\infty \frac{z^k}{(j\pi)^{k+1}} + \sum_{k=0}^\infty (-1)^k \frac{z^k}{(j\pi)^{k+1}} =$$

$$= -2\sum_1^\infty \frac{z^{2m-1}}{(j\pi)^{2m}} \qquad (|z|<\pi j).$$

Hence, the Taylor coefficients of even powers of z in the expansion of the function $\cot z - \frac{1}{z}$ are equal to zero (this follows immediately from the oddness of this function) and the coefficients of odd powers, z^{2m-1}, can be written in the form of the series

$$-2\sum_{j=1}^\infty \frac{1}{(j\pi)^{2m}} = -\frac{2}{\pi^{2m}}\sum_{j=1}^\infty \frac{1}{j^{2m}} \qquad (m=1, 2, \ldots).$$

Consequently

$$\cot z - \frac{1}{z} = \sum_{m=1}^\infty \left[-\frac{2}{\pi^{2m}}\sum_{j=1}^\infty \frac{1}{j^{2m}}\right] z^{2m-1}.$$

The same expansion was obtained in Art. 8 of Chapter 6 in a different form, namely

$$\cot z - \frac{1}{z} = \sum_{m=1}^{\infty} (-1)^m \frac{2^{2m} B_{2m}}{(2m)!} z^{2m-1}.$$

Comparing the two expansions, we obtain

$$\frac{2}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}} = (-1)^{m-1} \frac{2^{2m} B_{2m}}{(2m)!}.$$

Since the left-hand side of this equality is positive, the right-hand side must also be so, i.e. $(-1)^{m-1} B_{2m} > 0$. Hence, it follows that the Bernoulli numbers B_{2m} $(m=1, 2, 3, \ldots)$ must have alternating signs (in Art. 10 of Chapter 6, we did actually find that $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}, \ldots$).

From the foregoing relation, one readily obtains, in particular, the following equalities :

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{B_2}{2 \cdot 2!} (2\pi)^2 = \frac{\pi^2}{6},$$

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = -\frac{B_4}{2 \cdot 4!} (2\pi)^4 = \frac{\pi^4}{90},$$

$$\sum_{j=1}^{\infty} \frac{1}{j^6} = \frac{B_6}{2 \cdot 6!} (2\pi)^6 = \frac{\pi^6}{945},$$

. . . . . .

9

ANALYTIC CONTINUATION. THE CONCEPT OF A RIEMANN SURFACE. SINGULARITIES

1. ANALYTIC CONTINUATION

The problem of the *analytic continuation* of a function $f(z)$ defined on some set E consists in extending the definition of this function to a domain $D \supset E$, as wide as possible, so that $f(z)$ would be analytic in the domain D as well. The simplest examples of analytic continuation are provided by a transition from the functions of real variable e^x, $\sin x$, $\cos x$ (i.e. functions defined only on the real axis E) to the functions e^z, $\sin z$, $\cos z$ of a complex variable, analytic in the whole plane and coinciding on E with the corresponding function of the real variable. Such a transition can be effected by replacing in the power series

$$e^x = \sum_0^\infty \frac{x^n}{n!}, \quad \sin x = \sum_1^\infty (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, \quad \cos x = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

the real variable x by the complex variable z and by ensuring that these series would still remain convergent.

We futher examine the case of the power series $\sum_0^\infty z^n$, convergent in the unit circle $E: |z| < 1$. In this circle, it defines the analytic function $f(z) = \sum_0^\infty z^n = \frac{1}{1-z}$. Though the series diverges outside the unit circle, the function $f(z)$ can be analytically continued to a wider domain D representing the whole plane, excluding the point $z = 1$. It is sufficient to put $f(z) = \frac{1}{1-z}$ in this domain.

We now turn our attention from these particular cases to the general case. We first assume that the possibility of the continuation of $f(z)$ into the domain D as a one-valued and analytic function has already been

established. The problem now is to find a method for calculating its values in this domain. Such a method is, evidently, in conformity with the uniqueness theorem discussed in Art. 6 of Chapter 6. We shall now examine it in detail.

Let $E \subset D$ be a point set having at least one limit point z_0 in the domain D. We start with the determination of the coefficients of the power series

$$f(z) = c_0^{(0)} + c_1^{(0)}(z-z_0) + \ldots + c_n^{(0)}(z-z_0)^n + \ldots, \qquad (1)$$

representing $f(z)$ in the neighbourhood of the point z_0. It is well known that this series converges in the circle $K_0 : |z-z_0| < r_0$, where r_0 is the distance of the point z_0 from the boundary Δ of the domain D (but the series may also converge in a larger circle). Let $\{z_k\}$ be a sequence of points, different from z_0, and from one another and convergent to z_0, in the set E. The values $f(z_k)$ $(k=1, 2, 3, \ldots)$ will be taken as known.

For $c_0^{(0)} = f(z_0)$, we evidently obtain

$$c_0^{(0)} = \lim_{z_k \to z_0} f(z_k).$$

We further assume that the coefficients $c_0^{(0)}, c_1^{(0)}, \ldots, c_{n-1}^{(0)}$ have already been calculated. Then, from the formula (1), we obtain for $c_n^{(0)}$ the expression

$$c_n^{(0)} = \lim_{z_k \to z_0} \frac{f(z_k) - c_0^{(0)} - c_1^{(0)}(z_k-z_0) - \ldots - c_{n-1}^{(0)}(z_k-z_0)^{n-1}}{(z_k-z_0)^n}.$$

Similarly, we can calculate one by one all the coefficients of the series (1). Now, let z' be an arbitrary point in the domain, not belonging to the circle K_0. Connecting z' with z_0 by a broken line $L \subset D$, we denote the distance between L and Δ by the symbol $\delta > 0$. We now divide L into arcs with lengths less than δ and let the points of division, taken in the direction from z_0 to z', be $z_0, z_1, z_2, \ldots, z_{m-1}, z_m = z'$. If r_j is the distance from the point z_j to Δ, then $f(z)$ can be represented, in the circle $K_j : |z-z_j| < r_j$, by the series

$$f(z) = c_0^{(j)} + c_1^{(j)}(z-z_j) + \ldots + c_n^{(j)}(z-z_j)^n + \ldots . \qquad (2)$$

The coefficients of the series have already been found for $j = 0$. We now assume that these have been calculated for the series (2) for any value $j < m$. Noting that the distance between the centres z_j and z_{j+1} of the circles K_j and K_{j+1} is less than the length of the corresponding

arc $\delta \leqslant r_j$, we find that z_{j+1} belongs to K_j. Hence, for calculating the coefficients $c_n^{(j+1)} = \frac{f^{(n)}(z_{j+1})}{n!}$ we can make use of the expansion of the function in the circle K_j. We thus obtain

$$c_n^{(j+1)} = \frac{f^{(n)}(z_{j+1})}{n!} = c_n^{(j)} + \frac{(n+1)}{1!} c_{n+1}^{(j)} (z_{j+1}-z_j) + \\ + \frac{(n+2)(n+1)}{2!} c_{n+2}^{(j)} (z_{j+1}-z_j)^2 + \ldots .$$

Similarly, the coefficients of every series subsequent to (2) can be expressed in terms of the coefficients of the preceding series. Hence, all the coefficients of the series corresponding to $j = m$ can be calculated —in particular $c_0^{(m)} = f(z_m) = f(z')$. Thus, the proposition set is completely proved. The reader will note that we have repeated almost without any change the procedure of the proof of the uniqueness theorem. The essential role in the solution of the problem has been played by the chain of circles K_j, in each of which a one-valued analytic function $f_j(z)$ [the sum of the power series (2)] is defined, every circle K_{j+1} having a part in common with the previous circle K_j and the values of the functions $f_{j+1}(z)$ and $f_j(z)$ coinciding in this common part. In the following article we shall discuss a generalization of this process. In this generalization the circles will be replaced by arbitrary (convex) domains.

2. DIRECT ANALYTIC CONTINUATION

We recall that the domain G is said to be *convex* if a rectilinear segment, connecting any two points of the domain, is contained within the domain itself. Examples of convex domains are the circle, the triangle, the rectangle, the half-plane, the angle (with aperture not greater than π), a strip included between parallel straight lines, etc. If two convex domains have a nonempty intersection (that is, they have at least one point in common), then this intersection itself is a convex domain (prove this!).

Let G be a convex domain and $f(z)$ a function, analytic and one-valued in this domain. The totality of the various series for the function, together with the domain, constitute the *elements of the analytic function* (or simply, the *elements*). We shall denote an element by the symbol $\{G, f(z)\}$.[1] Two elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ can be regarded

[1] The condition of convexity ensures the connectedness of the part common to the two domains. If we remove this condition, we can preserve the entire subsequent exposition, by introducing in the formulation more refinements necessitated by the fact that in the general case the part common to the two domains consists of several domains, any pair of which does not have common points.

as identical if, and only if, the domains G_1 and G_2 coincide and $f_1(z) = f_2(z)$ at all the points of the domain. Two elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$, satisfying the conditions: (1) that G_1 and G_2 have a nonempty intersection, and (2) that the values of $f_1(z)$ and $f_2(z)$ coincide in the part common to the two domains, are referred to as the *direct analytic continuation of each other.*

The elements $\{G_1, f_1(z)\}, \{G_2, f_2(z)\}, \ldots, \{G_n, f_n(z)\}$ constitute a *chain* of (direct) analytic continuation if every subsequent element $\{G_{j+1}, f_{j+1}(z)\}$ is the direct continuation of the preceding one. The chain joins the first element $\{G_1, f_1(z)\}$ with the last element $\{G_n, f_n(z)\}$. Obviously, it is the same chain which when traversed in the opposite direction joins the element $\{G_n, f_n(z)\}$ with $\{G_1, f_1(z)\}$. Two elements $\{G, f(z)\}$ and $\{D, \varphi(z)\}$ are called *analytic continuations* of each other if there exists a chain which connects one with the other.

Example 1. We now turn to the problem of Art. 1. Obviously, for any j ($j=0, 1, 2, \ldots, n$), the sum of the power series (2), along with the corresponding circle $K_j: |z-z_j| < r_j$, constitutes an element of an analytic function; we shall denote this element by the symbol E_j. Two neighbouring elements E_j and E_{j+1} are direct analytic continuations of each other. The elements $E_0, E_1, \ldots, E_n$ constitute a chain of analytic continuations connecting E_0 with E_n; E_0 and E_n (and in general, E_j and E_k, $0 \leqslant j \leqslant n$, $0 \leqslant k \leqslant n$) are analytic continuations of one another. Finally, the process by which the problem of Art. 1 was solved can be referred to as the *process of analytic continuation.*

From Art. 1 it is also evident that if for any point $\zeta \in D$ we form a power series

$$f(\zeta) + f'(\zeta)(z-\zeta) + \ldots + \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n + \ldots,$$

converging to $f(z)$ in the circle $K(\zeta): |z-\zeta| < \rho$, where ρ is the distance from the point ζ to Δ, then we obtain an infinite set of elements $\{K(\zeta), f(z)\}$, of which each one is an analytic continuation of the other, while all of them together can be used to determine (or define) the given one-valued function $f(z)$ in the domain D.

Example 2. Let G_j be a half-plane, defined by the following inequalities for the polar angle φ: $j\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}$ ($j=0, \pm 1, \pm 2 \ldots$). Evidently, $G_{j+4n} = G_j$ (n being an integer). We put $f_j(z) = \ln|z| + i\varphi$, where φ satisfies the same inequalities, and obtain the element $\{G_j, f_j(z)\}$. It can be easily seen that, for any integral j, the elements $\{G_j, f_j(z)\}$ and

$\{G_{j+1}, f_{j+1}(z)\}$ are direct analytic continuations of each other (the part common to the half-planes G_j and G_{j+1} being the quadrant $(j+1)\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}$, in which the functions f_j and f_{j+1} possess the same values). Hence, it follows that the elements $\{G_j, f_j\}$ and $\{G_k, f_k\}$ are analytic continuations of each other. Obviously, the totality of all the elements $\{G_j, f_j(z)\}$ can be used to define the many-valued function Ln z.

3. THE CONSTRUCTION OF AN ANALYTIC FUNCTION FROM ITS ELEMENTS

The set M of the elements $\{G, f(z)\}$ [finite or infinite] will, for the sake of brevity, be called *connected* if any two elements of this set, say $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$, are analytic continuations of each other and if the given set, for every pair of its elements, also contains elements of some chain connecting the elements of the pair. For example, the sets of elements in the Examples 1 and 2 of Art. 2 are connected. We shall now show that the union $\cup G = D$ of all the domains, appearing in the definition of the elements of a connected set M, is itself a domain. In fact, if $z \in D$, then z belongs at least to one of the domains G and, consequently, the neighbourhood of the point z, contained in G, is also contained in D. Let z_0 and z^* be two different points of the set D and let z_0 belong to the domain G_0 with a certain element $\{G_0, f_j(z)\} \subset M$ and let the point z^* belong to the domain G^* with a certain element $\{G^*, f^*(z)\}$. Further, let $\{G_0, f_0(z)\}$, $\{G_1, f_1(z)\}, \ldots, \{G_n, f_n(z)\}$ [$f^* = f_n$ and $G^* = G_n$] be a chain of elements of the set M connecting $\{G_0, f_0(z)\}$ with $\{G^*, f^*(z)\}$. We now select points z_{j+1}, in each of the intersections of the domains G_j and G_{j+1} $(j=0, 1, \ldots, n-1)$. Then, z_0 can be joined with the point z_1 by a rectilinear segment $\delta_1 \subset G_0$ (consequently, $\delta_1 \subset D$), z_1 can be connected with z_2 by a rectilinear segment $\delta_2 \subset G_1 \subset D, \ldots, z_{n-1}$ can be connected with $z_n = z^*$ by a rectilinear segment $\delta_n \subset G_{n-1} \subset D$. Obviously, the totality of these rectilinear segments $\delta_1, \delta_2, \ldots, \delta_n$ form a broken curve, contained in D and connecting the point z_0 with z^*. Thus, D is a domain. We can now define an analytic function $f(z)$ (in general, many-valued) in D with the help of the set M in the following manner.

If $z_0 \in D$, then selecting one of the elements $\{G_0, f_0(z)\}$, for which $z_0 \in G_0$, we assume $f(z) = f_0(z)$ and, in particular, $f(z_0) = f_0(z_0)$ in the neighbourhood of the point z_0 contained in G_0. We shall thus obtain one of the one-valued branches of the analytic function $f(z)$ in the given neighbourhood of the point z_0. Since there can be several (even an

infinite number of) different elements $\{\tilde{G}, \tilde{f}(z)\}$ for which $z_0 \in \tilde{G}$, we shall obtain several (even an infinite number of) neighbourhoods of the point z_0 and a corresponding one-valued branch of the function $f(z)$ in each of these neighbourhoods. Evidently, if we were to fix any one of them and also a definite value $f(z_0)$ along with it, we would find it necessary to determine not only the point z_0 but also one definite element $\{\tilde{G}, \tilde{f}(z)\} \in M$, satisfying the condition $z_0 \in \tilde{G}$. In this case, we shall obtain a one-valued branch of the function $f(z)$, not only in the circular neighbourhood of the point z_0 but also in the entire domain $\tilde{G}$. This branch will be defined by the function $\tilde{f}(z)$. In future, we shall look upon each element $\{\tilde{G}, \tilde{f}(z)\}$ as an element of the function $f(z)$ constructed here. For the purpose of illustration, we turn our attention to the sets of elements in the Examples 1 and 2 of Art. 2. In the first of these examples, a union of the circles $K(\zeta)$ gives us the original domain D and the function $f(z)$ defined by its elements $\{K(\zeta), f(\zeta)+f'(\zeta)(z-\zeta)+\ldots\}$ is found to be one-valued and analytic in the domain D of the functions. In the the second case, the union of all the half-planes G_j: $j\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}$ $(j=0, \pm 1, \pm 2 \ldots)$ represents the whole plane, excluding the points $z = 0$ and $z = +\infty$. The elements $\left\{ j\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}, \ \ln|z| + i\varphi \right\}$, considered above, define in this domain an infinite-valued analytic function, viz. Ln z. In each one of the four half-planes: $j_0\frac{\pi}{2} < \varphi < (j_0+2)\frac{\pi}{2}$ $(j_0=0, 1, 2, 3)$ [the infinite set of the half-planes G_j, for which $j = 4n + j_0$, $n = 0 \pm 1, \pm 2, \ldots$, coincides with such a half-plane], Ln z possesses an infinite number of one-valued branches, given by the formulae

$$f_{4n+j_0}(z) = \ln|z| + i\varphi + i2\pi n.$$

4. CONSTRUCTION OF A RIEMANN SURFACE

We owe to Riemann the generalization of the concept of a domain in such a way that any many-valued analytic function $f(z)$ of a complex variable z becomes one-valued if it is regarded as a function of points of the corresponding generalized domain.

Let M be a connected set of elements $\{G, f(z)\}$. When we constructed the domain $D = \cup G$, any point z belonging to the domain G_0 of some element $\{G_0, f_0(z)\}$ and also to the domain G^* of some element $\{G^*, f^*(z)\}$ was regarded not as two points but as one and the same point of the domain D.

We shall now modify the process of union, described above, of the domains $\{G\}$ and regard the point z belonging to the domains G_0 and G^* of the elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ as one and the same point if, and only if, these elements are direct analytical continuations of each other (that is, if the values of $f_0(z)$ and $f^*(z)$ coincide in the intersection of G_0 and G^*). To visualize this process, we imagine that for every element $\{G, f(z)\}$ we have a model prepared for the corresponding domain G, made of a piece of paper or textile, with corresponding outlines. The process of union of the domains $\{G\}$ will be represented by pasting these pieces together, along with those parts whose points correspond identically. In other words, the pieces representing the domains G_0 and G^* with the elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ are pasted to each other only when the following two conditions are simultaneously satisfied: (1) the domains G_0 and G^* possess a non-empty intersection $G_0 \cap G^* = g$; (2) $f_0(z) = f^*(z)$ at all the points of g. When all possible pastings of the domain $\{G\}$ with one another have been effected according to this rule, we would obtain the *generalized domain* R which, generally speaking, is a multi-layered or a multisheeted domain over the domain D. However, in special cases, as in the Example 1 of Art. 2, the process of construction of R does not differ in any way from that of the construction of D, because whenever the circles $K(\zeta_0)$ and $K(\zeta^*)$ have a nonempty intersection, the sums of the corresponding power series (2) will be identical. So, in this case, the generalized domain R is identical with the domain D. In the Example 2 of Art. 2, the position is different. In this case, the domains G_j: $j\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}$, for all integral numbers j which leave one and the same remainder j_0 when divided by 4, represent one and the same half-plane $j_0\frac{\pi}{2} < \varphi < (j_0+2)\frac{\pi}{2}$, bounded by the corresponding coordinate axis. We shall now represent these domains in the form of an infinite set of paper sheets different from one another but superimposed in a single pile over the half-plane G_{j_0}. Since j_0 takes up the four values: 0, 1, 2 and 3, we obtain in all four such piles. No two sheets G_{4k+j_0} and G_{4l+j_0} $(k \neq l)$ of one and the same pile should be directly pasted with each other, because in their intersection (coinciding with the half-plane G_{j_0}) $f_{4k+j_0} = \ln|z| + i(\varphi+2k\pi)$, $f_{4l+j_0} = \ln|z| + i(\varphi+2l\pi)$ $\left(j_0\frac{\pi}{2} < \varphi < (j_0+2)\frac{\pi}{2}, k \neq l\right)$. Only those sheets that have their indices differing by unity, viz. G_j and G_{j+1} (they belong to two different piles), are pasted together along the

part common to them, located on the quadrant $(j+1)\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}$. In fact, the values of the functions $f_j(z) = \ln|z| + i\varphi\left(j\frac{\pi}{2} < \varphi < (j+2)\frac{\pi}{2}\right)$ and $f_{j+1}(z) = \ln|z| + i\varphi\left((j+1)\frac{\pi}{2} < \varphi < (j+3)\frac{\pi}{2}\right)$ coincide in this common part.

As a result of all these pastings, we obtain an infinite-sheeted domain R, located over the corresponding domain D (D is the z-plane, with the points $z = 0$ and $z = \infty$ excluded).

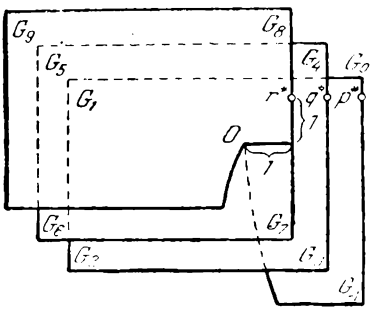


Fig. 51.

Fig. 51 gives a visual diagrammatic representation of the part R, obtained by pasting together the 11 half-planes G_{-1}, G_0, G_1, G_2, ..., G_9 (we propose that the reader should construct a model of 11 rectangular sheets of paper).

Obviously, for the definition of a point in the domain R, it is insufficient to give a complex number z, the affix of this point. Thus, in the last example, one and the same affix $1 + i$ has an infinite set of different points R, located on the various parts of R, obtained by pasting together G_{-1} with G_0, G_3 with G_4, G_7 with G_8, ..., in general G_{4k+3} with G_{4k+4} ($k = 0, \pm 1, \pm 2, \ldots$; see Fig. 51). In the general case, for defining a point $p \in R$, it is necessary to indicate, along with the affix z, a definite element $\{G^*, f^*(z)\}$ also, here $z \in G^*$. In this case, we obtain one definite value of the one-valued function at this point: $f(p) = f^*(z)$.

For instance, the value of the logarithm at the point p^*, marked on Fig. 51, is $\ln\sqrt{2} + i\frac{\pi}{4}$, at the point q^* it is $\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi\right)$ and at the point r^*, $\ln\sqrt{2} + i\left(\frac{\pi}{4} + 4\pi\right)$. Thus, in the generalized domain R, constructed by the above method, a given connected set of elements M: $\{G, f(z)\}$, determines a one-valued function. This function can be looked upon as analytic in the domain R. To ensure that the concept of analyticity can be directly transferred to functions in R, it is sufficient to note that the assignment of a point p^* on R implies the fixation of an affix z^* of this point and at the same time the definition of a definite element $\{G^*, f^*(z)\}$ in M, for which $z^* \in G^*$. If we proceed in this way, we can immediately select a part of R, located in the domain G^*, (this part is said to be projected onto the domain G^*) such that every point in this can be fully characterized by its affix $z \in G^*$ (the projection of the

point p). Thus, the function $f(p)$ of a point in the given part of R can be looked upon as a function of a complex variable z in the domain G and, consequently, we can apply to it all the concepts and results of the theory of functions of a complex variable. The generalised domain R described in this section is called the *Riemann surface* of the analytic function $f(z)$ [defined by means of the given connected set of elements M].

5. THE RIEMANN-SCHWARTZ PRINCIPLE OF SYMMETRY

The concept of a direct analytical continuation of two elements, $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$, with the overlapping domains provides the basis for the process of analytic continuation described above. It was essential in this connection that the single function $f(z)$, analytic in the greater domain $G = G_1 \cup G_2$ (Fig. 52), could be obtained with the help of the two functions $f_1(z)$ and $f_2(z)$ that are analytic in the two respective domains G_1 and G_2. In this case, $f(z)$ would coincide with $f_j(z)$ in G_j $(j=1, 2)$. It would be expedient to generalize the definition of an element and of direct analytic continuation as follows.

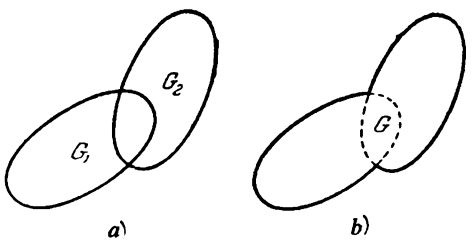


Fig. 52.

By an *element of an analytic function* we mean the aggregate consisting of the function $f(z)$, one-valued and analytic in the domain G bounded by a generalized Jordan curve Γ (the domain is not necessarily convex), and of the domain G itself. The element will, as before, be denoted by the symbol $\{G, f(z)\}$. Let $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ be two elements, the domains G_1 and G_2 having no point in common but their boundaries Γ_1 and Γ_2 having in common an open arc δ (i.e. without considering the end points; Fig. 53*a*). If in the domain $G = G_1 + G_2 + \delta$ there exists an analytic function $f(z)$, coinciding with $f_j(z)$ in G_j $(j=1, 2)$, then the elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are called, as before, *direct analytic continuations of each other*. We also say that the function $f_1(z)$ is *analytically continued* from the domain G_1 through the arc δ into the domain G_2, and $f_2(z)$ is its analytic continu-

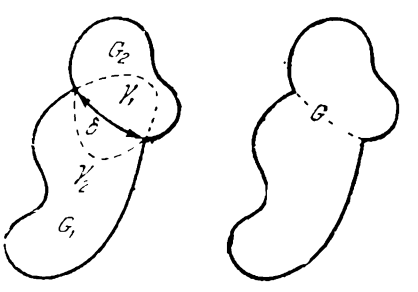


Fig. 53

ation [or $f_2(z)$ is analytically continued from the domain G_3 through the arc δ into the domain G_1, and $f_1(z)$ is its analytic continuation].

The case discussed here in which the domains *adjoin* one another can be reduced to the case examined earlier in which the domains *overlap*. Now, we connect the ends of the arc δ by arcs γ_1 and γ_2, as has been shown in Fig. 53*a*, and denote the curves obtained from Γ_1 and Γ_2 by replacing δ by γ_1 and γ_2, respectively by Γ_1' and Γ_2', and by G_1' and G_2' the interiors of Γ_1' and Γ_2'.

The domains G_1' and G_2' have a part in common, which is bounded by the Jordan curve $\gamma_1 \cup \gamma_2$. Evidently, when $f_1(z)$ is analytically continued from G_1 into G_2 through the arc δ, we can continue $f_1(z)$ analytically into the domain G_1' and $f_2(z)$ into the domain G_2'; in this case, the elements $\{G_1', f_1(z)\}$ and $\{G_2', f_2(z)\}$, the domains of which overlap, are direct analytic continuations of each other (in the previous sense of the word; see Art. 2, also footnote on page 285). Similarly, we can verify the truth of the converse statement that analytic continuation with the help of elements belonging to overlapping domains can always be replaced by an analytic continuation with the help of elements belonging to adjoining domains.

Starting from the direct analytic continuation with the help of elements belonging to overlapping domains, we can introduce the concept of a *chain of analytic continuations* (not direct), and, finally, arrive at the construction of a Riemann surface, as has been done in Arts. 2—4. In this process of construction of a Riemann surface, the two domains G_0 and G^* of elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ are connected (pasted on) to each other along the common boundary arc δ if, and only if, $f_0(z)$ is continued through δ into the domain G^*, $f^*(z)$ being the result of such a continuation. We recommend to the reader to try the process by which the Riemann surface of the function Ln z, obtained in Art. 4, by pasting together an infinite set of copies of four different half-planes (upper, lower, right and left) overlapping one another, may also be obtained by pasting together an infinite set of copies of only two different half-planes, say, the upper and the lower, alternately connected along the positive and negative parts of the real axis.

We shall now show that the elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ will be direct continuations of each other in the special case when δ is a segment of a straight line; the functions $f_j(z)$ are continuous on $G_j + \delta$ and take up equal values at the points of δ. In fact, assuming $f(z) = f_j(z)$, $z \in G_j$ $(j=1, 2)$ and $f(z) = f_1(z) = f_2(z)$ $(z \in \delta)$, we find that the function $f(z)$ is continuous in the domain $G = G_1 \cup G_2 \cup \delta$ and

analytic in each of the domains G_j $(j=1, 2)$. Consequently, $\int_{\gamma} f(z)\,dz = 0$, where γ is any triangular contour lying wholly in G_1 or in G_2. Hence, it follows by Art. 4 of Chapter 6 that $f(z)$ is a function analytic in the domain G, i.e. the elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are analytic continuations of each other. Now it will be easy to prove the following important theorem.

The Riemann-Schwartz principle of symmetry. *Let G be a domain, bounded by the Jordan curve Γ, containing the rectilinear segment δ. Let there be defined on the set $G + \delta$ a continuous function $f(z)$ which is analytic in the domain G and takes up at the points of the segment δ values that lie on some straight line Λ. We construct a domain G^* which is symmetric to the domain G with respect to δ (more precisely, with respect to a straight line containing δ) and define on $G^* + \delta$ a function $f^*(z)$ such that $f^*(z) = f(z)$, when $z \in \delta$, and that $f^*(z)$ is symmetric to $f(z)$ with respect to Λ if the point $z^* \in G^*$ is symmetric to $z \in G$ with respect to δ. Then, the elements $\{G, f(z)\}$ and $\{G^*, f^*(z)\}$ will be direct analytic continuations of each other.*

This theorem gives us the simple and sufficient conditions under which the function $f(z)$, defined in the domain G, can be analytically

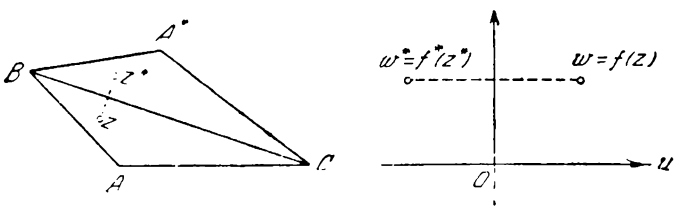


Fig. 54.

continued into the domain $G + \delta + G^*$ which is, so to say, 'doubly greater' than G, and also shows, at the same time, how this continuation can actually be effected (viz. by the principle of symmetry). Before we take up the proof of the theorem we explain it through a simple example. Let $w = f(z)$ be continuous in the closed triangle ABC (Fig. 54), analytic in its interior and let it take up on the segment BC purely imaginary values. Then, the given theorem enables us to state that the function $f(z)$ can be analytically continued into the domain ABA^*C. If, at the same time, the points z^* and z of the above mentioned domain are symmetric with respect to BC, then the values w^* and w of the continued function at these points will also be symmetric with respect to the imaginary axis (Fig. 54).

We now pass on to the proof. First of all, we subject the plane of z and the plane of w to linear transformations $z' = az + b$ and $w' = \alpha w + \beta$, choosing these transformations in such a way that the segment δ translate itself onto some segment of the real axis in the z'-plane and the straight line Λ translate itself into the real axis of the w'-plane. At the same time, the domains and functions referred to in the conditions of the theorem are accordingly changed {for example, we have the function $w' = \alpha f\left(\frac{z'-b}{a}\right) + \beta$ instead of the function $w = f(z)$}. We, however, preserve the previous notation for these; so also for the variables z and w. Essentially, the given transformations (and also the transformations inverse to them) translate a finite plane into a finite plane, a domain into a domain, a straight line into a straight line, preserve the symmetry of disposition of the points with respect to the straight line subjected to transformation, and, finally, preserve the continuity and analyticity of the functions. Hence, extending the proof to the case when the segment δ lies on the real axis and the straight line Λ coincides with the real axis, we shall also find that the theorem as formulated originally is again valid.

Further, at the points of the segment δ, $z = x$ (i.e $y = \operatorname{Im} z = 0$) and $w = f(z) = f(x) = u$ (i.e. $v = \operatorname{Im} w = 0$). Also the points z and z^* are symmetric with respect to δ and likewise the points w and w^* are symmetric with respect to Λ and are represented by a pair of mutually conjugate complex numbers. We now show that the function $f^*(z^*)$, as defined in the conditions of the theorem, is analytic in the domain G^* and continuous on $G^* + \delta$. Indeed, if $z_0^* \in G^*$ and if its neighbourhood $|z^* - z_0^*| < \rho$ are contained in G^*, the point $z_0 = \bar{z}_0^* \in G$ and its neighbourhood $|z - z_0| < \rho$ are also contained in G (Fig. 55).

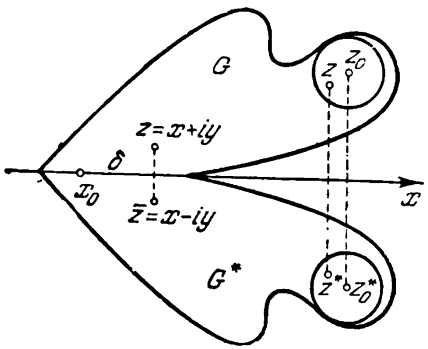


Fig. 55.

But $f(z)$ is represented in the neighbourhood of the point z_0 by the power series

$$f(z) = c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0)^n + \ldots .$$

Hence, at the symmetric point $z^* = \bar{z}$, belonging to the afore-mentioned neighbourhood of the point $z_0^* = \bar{z}_0$, we must have, by definition,

$$f^*(z^*) = f(\bar{z}) = \bar{c}_0 + \bar{c}_1(\bar{z}-\bar{z}_0) + \ldots + \bar{c}_n(\bar{z}-\bar{z}_0)^n + \ldots =$$
$$= \bar{c}_0 + c_1(z^*-z_0^*) + \ldots + \bar{c}_n(z^*-z_0^*)^n + \ldots .$$

We find that $f^*(z^*)$ can be represented by a convergent power series in the neighbourhood of any arbitrary point $z_0^* \in G^*$; hence, it is an analytic function.

It remains to show that $f^*(z^*)$ is continuous in $G^* + \delta$. For proving this we only require that it be continuous at every point $x_0 \in \delta$.

Let $y > 0$ in a half-neighbourhood of the point x_0, belonging to the domain G. By the condition of the theorem : $\lim\limits_{z \to x_0,\ z \in G} f(z) =$ $= \lim\limits_{x \to x_0,\ y \to 0\ (y>0)} [u(x, y) + iv(x, y)] = f(x_0) = u(x_0, 0)$ and, hence,

$$\lim_{x \to x_0,\ y \to 0\ (y>0)} [u(x, y) - iv(x, y)] = f(x_0) = u(x_0, 0);$$

putting $z^* = \bar{z}$, where $z = x + iy \in G$, and noting that $z^* \to x_0$ implies $z = \bar{z}^* \to x_0$, we obtain

$$\lim_{z^* \to x_0,\ z^* \in G^*} f^*(z^*) = \lim_{z \to x_0,\ z \in G} \overline{f(z)} =$$
$$= \lim_{x \to x_0,\ y \to 0\ (y>0)} [u(x, y) - iv(x, y)] = f(x_0) = f^*(x_0).$$

Thus, the continuity of $f^*(z^*)$ in the set $G^* + \delta$ is proved. It now remains to note that the elements $\{G, f(z)\}$ and $\{G^*, f^*(z)\}$ satisfy the conditions formulated on page 293; consequently, they are direct analytic continuations of each other.

Thus, the theorem is proved.

6. SINGULARITIES ON THE BOUNDARY OF THE CIRCLE OF CONVERGENCE OF A POWER SERIES

We now turn to explain the concept of a singularity. We recall (Art. 3, Chapter 7) that while examining the function $f(z)$, which is one-valued and analytic in the domain $D : 0 < |z-z_0| < R$, we called the point z_0 an isolated boundary point of the domain D, a regular or a singular point of the function $f(z)$, depending upon whether we can or cannot define the function at the point z_0, so that it be analytic in the entire circle $|z-z_0| < R$ (including the point z_0). The concept of an

isolated singularity of a simple character was introduced in the same way.

Here, we shall give the method of extending the concept of a regular and a singular point to nonisolated boundary points of a domain. Let G be a domain bounded by a closed Jordan curve Γ and $f(z)$ be one-valued and analytic in this domain. Then, $f(z)$ and G define the element $\{G, f(z)\}$. We shall call the point $\zeta \in \Gamma$ a *regular point* of the element $\{G, f(z)\}$ [in short, a *regular point* of the function $f(z)$], if there exists a neighbourhood $U_\zeta : |z-\zeta| < \rho$, of the point ζ and if there be a one-valued analytic function $\varphi_\zeta(z)$, coinciding with $f(z)$ in the part common to this neighbourhood and the domain G. In this case, there, evidently, exists an arc δ of the curve Γ, containing the point ζ, through which $f(z)$ can be analytically continued (Fig. 56).

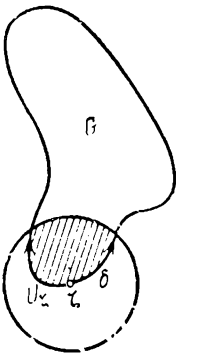


Fig. 56.

The boundary point ζ which is not a regular point is a *singular point of the element* $\{G, f(z)\}$ [in short, the *singular point of* $f(z)$]. In this case, in any neighbourhood of the point ζ, there cannot be any one-valued analytic function which would coincide with $f(z)$ in the part common to this neighbourhood and the domain G. In other words, there does not exist any arc $\delta \subset \Gamma$ containing the point ζ within it, through which the function $f(z)$ might be analytically continued from the domain G.

It follows from the definition of a regular point that if $\zeta_0 \in \Gamma$ be such a point, then all the points of some arc δ, containing this point, are also regular points. Hence, in particular, it follows that if there exists a single regular point, then there also exists an infinite set of such points. Contrary to this, a singular point can be a unique point.

We shall find that any point $z_0 \in G$ possesses the characteristic property of a regular point, i.e. for it there would exist a neighbourhood K_0 (we can take any neighbourhood of the point z_0, contained in G) and a function analytic in K_0 [the function $f(z_0)$ itself] coinciding with $f(z)$ at all points common to K_0 and G, i.e. in K_0. Hence, all the points of the domain G will be regular points of the element $\{G, f(z)\}$.

Example. Let G be a unit circle and $f(z) = \frac{1}{1-z}$. For every point ζ lying on the circle $\Gamma : |z| = 1$ and being different from unity, there exists a neighbourhood K: $|z-\zeta| < |1-\zeta|$, and a function $\varphi(z) = \frac{1}{1-z}$, analytic in it and coinciding with $f(z)$ at the points

common to K and G. Then, every such point ζ will be a regular point of $\frac{1}{1-z}$. We shall show that $\zeta_0 = 1$ is a singularity.

Conversely, in some neighbourhood K_0 of this point, there will exist an analytic function $\varphi(z)$, coinciding with $\frac{1}{1-z}$ in the part common to K_0 and G. Then, there would exist the finite limit

$$\lim_{z \to 1 \; z \in K \cap G} \varphi(z) = \lim_{z \to 1} \frac{1}{1-z} = \varphi(1).$$

But, evidently, it cannot be so. Hence, the point 1 is a singularity of $\frac{1}{1-z}$ (that is, a pole).

Theorem. *Let* $\sum_0^{\infty} a_n (z-z_0)^n = f(z)$ *be a power series with a finite radius of convergence* R. *Then, on the boundary* Γ *of the circle of convergence* K, *there exists at least one singularity for the sum of this power series.*

We first assume that the theorem is not true. Then, any point ζ on the circle Γ must be a regular point of $f(z)$. We construct for any point ζ a neighbourhood K_ζ and a function $\varphi_\zeta(z)$ which is analytic in it and coincides with $f(z)$ in the part common to K_ζ and K. The union of the circle K and all the circles K_ζ represents a domain D into which $f(z)$ can be analytically continued. We shall now show that the function obtained as a result of such a continuation will be one-valued.

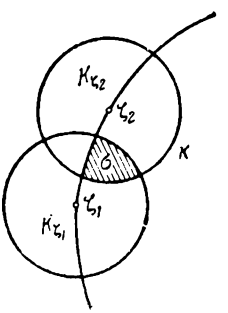


Fig. 57.

Let the circles K_{ζ_1} and K_{ζ_2} have a part in common — a lune (Fig. 57). Then, there must exist a part σ of the lune which is common to the three circles K, K_{ζ_1} and K_{ζ_2} (this portion is shaded in the figure). Since the values of any of the functions $\varphi_{\zeta_1}(z)$ and $\varphi_{\zeta_2}(z)$ coincide in σ with the values of $f(z)$, we have $\varphi_{\zeta_1}(z) = \varphi_{\zeta_2}(z)$ $(z \in \sigma)$ and, by the uniqueness theorem, $\varphi_{\zeta_1}(z) = \varphi_{\zeta_2}(z)$ at all the points of the lune. Thus, if some point of the domain D belongs to two different circles K_{ζ_1} and K_{ζ_2} then, taking in it the value of the continued function $\varphi_{\zeta_1}(z)$ or $\varphi_{\zeta_2}(z)$, we obtain one and the same number. Applying this argument to each pair of circles belonging to the set $\{K_\zeta\}$, we establish the one-valuedness of the continued function $f(z)$. We further note that any point $\zeta \in \Gamma$ is an interior point of the

domain D ; hence, the distance R' of the point z_0 to the boundary Δ of the domain D will be greater than R. But then the power series representing $f(z)$ and expanded in powers of $z - z_0$ (obviously, it cannot differ from the given series) must converge in the circle $|z-z_0| < R'$, with a radius greater than R ; this, however, contradicts the conditions of the theorem (R being the radius of convergence of the given series). This proves the theorem.

Corollary. *In order that the radius of convergence R of the Taylor expansion of the function $f(z)$:*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \ldots + \frac{f^n(z_0)}{n!}(z-z_0)^n + \ldots, \qquad (3)$$

analytic in some circle $K: |z-z_0| < \rho$, be the same as the radius of this circle, it is necessary and sufficient that there should exist on the circle $\Gamma : |z-z_0| = \rho$, at least one singularity of the element $\{K, f(z)\}$.

In fact, if $R = \rho$ then, by the theorem proved above, we find that at least one singularity of the element lies on the circle Γ. Hence, the condition is necessary for the equality of R and ρ. And it is also sufficient for this purpose. Now, by the theorem of Art. 2 of Chapter 6, $R \geqslant \rho$. If we assume that at least one singularity of the element lies on the circle Γ and $R \neq \rho$, then we must have $R > \rho$. In this case, the sum of the power series (3) represents a function analytic in the circle $|z-z_0| < R$ and, hence, analytic in some neighbourhood of every point of the circle Γ, coinciding with $f(z)$ in the interior of Γ. Thus, it follows that every point of the circle Γ is a regular point of $f(z)$. From the contradiction thus arrived at, the validity of our statement follows.

As an example, we consider the geometric series

$$1 + z + z^2 + \ldots + z^n + \ldots .$$

Its radius of convergence is equal to unity and its sum is $\frac{1}{1-z}$. As we have already seen, there is actually a singularity here and that also a unique one, viz. $z = 1$, on the boundary of the circle of convergence $|z| = 1$.

We can easily give examples of power series for the sum of which every point on the boundary of the circle of convergence is a singularity. Here is one of the simplest examples of this kind.

Let us examine the series

$$f(z) = 1 + z^2 + z^4 + \ldots + z^{2^n} + \ldots .$$

Its radius of convergence is, evidently, equal to unity. We shall show that as z tends to unity from within the unit circle along its radius (that is, along the real axis), $f(z)$ tends to ∞. Now, for any natural number n, the partial sum of the series $1 + x^2 + \ldots + x^{2^n}$ tends to $n + 1$ as $x \to 1$; consequently, it satisfies the inequality

$$1 + x^2 + \ldots + x^{2^n} > n \quad \text{for } 1 - x < \delta(n), \text{ i.e. } x > 1 - \delta(n).$$

But, for the same values of x, we have

$$f(x) = \sum_0^\infty x^{2^k} > \sum_0^n x^{2^k} > n;$$

hence, it follows that

$$\lim_{x \to 1} f(x) = \infty.$$

On the basis of this fact we can easily establish, as in the earlier case of the geometric series, that the point 1 is a singularity of $f(z)$.

We now write the identity

$$f(z) = z^2 + z^4 + \ldots + z^{2^n} + [1 + (z^{2^n})^2 + (z^{2^n})^4 + \ldots].$$

Since the series in the square brackets differs from the original one only in that here z is replaced by z^{2^n}, we conclude that

$$f(z) = z^2 + z^4 + \ldots + z^{2^n} + f(z^{2^n}),$$

for any natural number n.

We now examine all the 2^n roots: $\sqrt[2^n]{1}$. These represent points located on a unit circle at the vertices of a regular 2^n-gon. If ζ is any one of them and the point z of the unit circle lies on the radius $O\zeta$, z^{2^n} obviously lies on the radius $O1$, and as $z \to \zeta$, $z^{2^n} \to 1$. Hence, it follows that

$$\lim_{\substack{z \to \zeta \\ z \in O\zeta}} f(z^{2^n}) = \infty$$

and, consequently,

$$\lim_{\substack{z \to \zeta \\ z \in O\zeta}} f(z) = \lim_{\substack{z \to \zeta \\ z \in O\zeta}} [z^2 + z^4 + \ldots + z^{2^n} + f(z^{2^n})] = \infty.$$

Thus, each of the roots $\sqrt[2^n]{1}$ is also a singularity of $f(z)$ [for all $n = 1, 2, 3 \ldots$]. We thus find that the set of the singularities of $f(z)$ is everywhere dense on the unit circle (so that any arc, however small, of the circle contains points belonging to this set).

It thus follows that all the points, without exception, of the unit circle are singularities of $f(z)$, because for a regular point, if it is situated on the circle, there would exist an arc, all the points of which must be regular. In the present case, however, this is impossible.

7. CRITERION FOR LOCATING SINGULARITIES

We now indicate a general method which would enable us to decide for any point ζ, lying on the boundary Γ of the circle of convergence of the power series (3), whether it would be a regular or a singular point for the sum of the series $f(z)$. Let z_1 be a point on the radius $z_0\zeta$, being different from z_0 and ζ. We expand $f(z)$ into a power series in $z - z_1$. We thus obtain

$$f(z) = b_0 + b_1 (z-z_1) + \ldots + b_n (z-z_1)^n + \ldots, \tag{4}$$

where

$$b_n = \frac{f^{(n)}(z_1)}{n!} = a_n + \frac{n+1}{1} a_{n+1} (z_1-z_0) + \\ + \frac{(n+1)(n+2)}{1 \cdot 2} a_{n+2} (z_1-z_0)^2 + \ldots \quad (n=0, 1, 2, \ldots).$$

According to the theorem of Art. 2 of Chapter 6, the above series converges in the circle $| z-z_1 | < \Delta$, where Δ is the distance from z_1 to Γ, i.e. $\Delta = R - | z_1-z_0 |$.

Thus, the series (4) converges in the interior of the circle γ, with its centre at the point z_1 and touching the circle Γ at the point ζ. By the Cauchy-Hadamard formula, the radius of convergence of the series (4) is

$$r = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{| b_n |}}.$$

If r coincides with Δ, then, on the circle $\gamma: |z-z_0| = \Delta$, there must lie at least one singularity of the sum of the series (4). But no point $\zeta' \in \gamma$, lying in the interior of K, can be a singular point for this series, because in the neighbourhood of the point ζ', which belongs wholly to K, $f(z)$ is an analytic function, coinciding with the sum of the series (4) in the interior of γ. Consequently, the point ζ is a singular point. Obviously, it must be a singular point of $f(z)$ also, which is the sum of the series (3). Assuming the contrary, we have a function $\varphi(z)$ analytic in some neighbourhood of the point ζ and coinciding with $f(z)$ at those points of the neighbourhood which lie in the interior of K. Then, this function

would also coincide with the sum of the series (4) at those points of the neighbourhood which lie in the interior of γ, i.e. ζ would not be a singular point of the series (4).

Thus, if

$$\Delta = R - |z_1 - z_0| = r = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}},$$

the point ζ is a singularity of $f(z)$. We shall now show that when $\Delta \neq r$, i.e. $\Delta < r$, the point ζ is a regular point of $f(z)$.

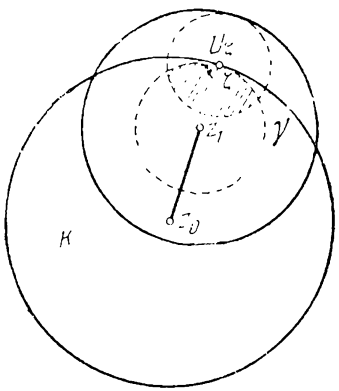


Fig. 58.

In fact, the sum of the series (4) represents in this case the function $\varphi(z)$, which is analytic in the neighbourhood U_ζ of the point ζ and coincides with $f(z)$ in the part of the circle U_ζ lying in the interior of γ (Fig. 58). But $f(z)$ and $\varphi(z)$ are one-valued analytic functions in the lune which is the common part of the circles U_ζ and K. Since they coincide in the shaded part of the lune, it follows from the uniqueness theorem that $\varphi(z)$ coincides with $f(z)$ in the whole of the lune. Thus, the point ζ is, in this case, a regular point of $f(z)$.

We have thus established that ζ *will be either a singular point or a regular point of* $f(z)$, *depending upon whether the equality*

$$\Delta = R - |z_1 - z_0| = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}} = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}}$$

or the inequality

$$\Delta < \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}}$$

holds.

As an illustration of this criterion, we shall prove the following theorem due to Pringsheim.

Theorem. *If the coefficients of the series* $\sum_0^\infty a_n z^n$, *with the unit circle of convergence, be real, nonnegative numbers* $a_n \geqslant 0$, *then the point* $z = 1$ *is a singular point of the sum of the series.*

To prove this, we take any point $z_1 = x$ on the radius $O1$. If we assume that the point 1 is not a singular point of the sum of the series, then, by what has been proved above, the inequality

$$\Delta = R - |z_1 - z_0| = 1 - x < \frac{1}{\overline{\lim}\sqrt[n]{\frac{|f^{(n)}(x)|}{n!}}} \tag{5}$$

must be satisfied.

We now examine an arbitrary point ζ on the unit circle. Let z_1 be a point, on the radius $O\zeta$, located on the circle $|z| = x$, i.e. $|z_1| = x$. Then, the distance Δ of z_1 from the unit circle will also be equal to $1-x$. On the other hand,

$$|f^{(n)}(z_1)| = \left| a_n + \frac{n+1}{1} a_{n+1} z_1 + \frac{(n+1)(n+2)}{2!} a_{n+2} z_1^2 + \dots \right| \leqslant$$

$$\leqslant a_n + \frac{n+1}{1} a_{n+1} x + \frac{(n+1)(n+2)}{2!} a_{n+2} x^2 + \dots = f^{(n)}(x)$$

and, consequently,

$$\frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}} \geqslant \frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{\frac{f^{(n)}(x)}{n!}}}. \tag{6}$$

Hence, for the point z_1 we have, by the inequalities (5) and (6),

$$\Delta < \frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}};$$

hence, it follows that for the sum of the series $\sum_0^\infty a_n z^n$ any point of the unit circle is a regular point; this, as we know, contradicts the conditions of the theorem to be proved (namely that the unit circle is the boundary of the circle of convergence).

Thus, the point $z = 1$ must be a singular point of the sum of the series $\sum_0^\infty a_n z^n$, provided that $a_n \geqslant 0$ and $R = 1$.

It can be seen from the proof of the theorem that if instead of the point 1 we take any other point ζ on the unit circle, then we would require that the numbers $a_n\zeta^n$ be nonnegative in order that ζ be a singular point. Moreover, it would be sufficient if these numbers were real and

nonnegative, starting only from a certain number $n \geqslant n_0$; this is so because by expressing $f(z)$ in the form

$$f(z) = \sum_0^{n_0-1} a_k z^k + \sum_{n_0}^{\infty} a_k z^k,$$

we can immediately see that the point ζ would be a singular point of $f(z)$ if, and only if, it is a singular point of the sum of the series $\sum_{n_0}^{\infty} a_k z^k$.

We now consider, as an example, the series $\sum_0^{\infty} \frac{z^{2^k}}{2^{k^2}}$. Here, the coefficients a_n are equal to zero if $n \neq 2^k$ and equal to $\frac{1}{2^{k^2}}$ if $n = 2^k$. Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \lim_{k\to\infty} \sqrt[2^k]{\frac{1}{2^{k^2}}} = 1,$$

whence it follows, by the Cauchy-Hadamard formula, that the radius of convergence of the series is unity.

Thus, by the theorem proved above, the point $z = 1$ is a singularity for the sum of the series $f(z)$. But, from the same theorem (according to the remark made above), it follows that any point $\zeta = \sqrt[n]{1}$, where n is an arbitrary natural number, is also a singularity of $f(z)$. Indeed, for $k \geqslant n$, we have

$$\frac{\zeta^{2^k}}{2^{k^2}} = \frac{(\zeta^{2^n})^{2^{k-n}}}{2^{k^2}} = \frac{1}{2^{k^2}} > 0.$$

Thus, the set of singularities of the function $f(z)$ is everywhere dense on the unit circle. It thus follows that there is no regular point of any element of $f(z)$ on this circle, i.e. all points ζ ($|\zeta| = 1$) are singular.

It is remarkable that this circumstance does not prevent the given power series from converging absolutely and uniformly in the closed unit circle and its sum $f(z)$ from being an infinitely differentiable function on the set $|z| \leqslant 1$ (especially, at the singularities). In fact, for $|z| \leqslant 1$, we have the inequality

$$\left|\frac{z^{2^k}}{2^{k^2}}\right| \leqslant \frac{1}{2^{k^2}},$$

and since the series $\sum_0^\infty \frac{1}{2^{k^2}}$ converges, the series $\sum_0^\infty \frac{z^{2^k}}{2^{k^2}}$ also converges absolutely and uniformly in the closed circle ; thus, its sum $f(z)$ is continuous in the closed circle. Further, if the given series is differentiated termwise m times, we would obtain the series

$$\sum_0^\infty \frac{2^k(2^k-1)\dots(2^k-m+1)}{2^{k^2}} z^{2^k-m},$$

the moduli of whose terms would satisfy, for $|z| \leqslant 1$, the inequalities

$$\left|\frac{2^k(2^k-1)\dots(2^k-m+1)}{2^{k^2}} z^{2^k-m}\right| \leqslant \frac{2^{km}}{2^{k^2}} = \frac{1}{2^{k(k-m)}} < \frac{1}{2^k}$$

for $k > m$.

Hence, the series obtained by a termwise differentiation of the power series $\sum_0^\infty \frac{z^{2^k}}{2^{k^2}}$ converges uniformly in the closed circle $|z| \leqslant 1$; from this, according to a well known theorem for the functions of a real variable and also applicable, without any change, to the functions of a complex variable, it follows that these series represent the derivatives $f^{(m)}(z)$. Thus, $f(z)$ is a function which is continuous and infinitely differentiable in the closed circle $|z| \leqslant 1$ and is also analytic in the interior of this circle ; moreover, for this function every point of the unit circle is a singularity.

This instructive example shows that the existence of singularities of an analytic function on the boundary of the domain in question (the circle) cannot, in some cases, outwardly affect the behaviour of the function near the singularity. Speaking more precisely, it does not manifest itself as a violation of the continuity of the function or of its derivatives at the given boundary point ζ. It is necessary to examine all the derivatives of $f(z)$ at some point z_1 of the given domain, so that, by comparing the quantity $\left[\overline{\lim}\sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}\right]^{-1}$ with the distance Δ from the point z_1 to the point ζ, we can determine whether the point ζ is a singular or a regular point.

8. DETERMINATION OF THE RADIUS OF CONVERGENCE OF A POWER SERIES FROM A KNOWN DISTRIBUTION OF SINGULARITIES OF THE FUNCTION

The proposition established on page 297 often enables us to determine the radius of convergence of the Taylor expansion of the analytic function

$f(z)$ without having to calculate the coefficients of the series, $\frac{f^{(n)}(z)}{n!}$. In fact, the whole problem reduces to finding the singularities of the elements. In this case, we should bear in mind that for the given function $f(z)$, which is analytic in a certain domain G, there exists an infinite set of different elements corresponding to the infinite set of circles, with different centres, belonging to the domain. Thus, we have to deal with the singularities of different elements of one and the same analytic function and it may so happen that a point which is a singularity of some elements is found to be regular for the others. We shall explain this with the help of simple examples. We first note that when the analytic function $f(z)$ is expressed by a formula containing a finite number of elementary functions, the possible singularities of its elements can easily be found among the points of discontinuity of this function, i.e. among those points at which this function becomes infinity and also among the branch points of the function $f(z)$.

Example 1. Let $f(z) = \frac{1}{1+z^2}$. This function is one-valued and analytic in the whole plane excluding the points $z = \pm i$ at which it becomes infinity. Let z_0 be an arbitrary point different from $\pm i$. We describe about z_0 as centre a circle $\gamma: |z-z_0| = \rho$, passing through the point $+i$ or $-i$ whichever is nearer to z_0. To be definite, let the nearer point be i. Then, $f(z)$ is analytic in the interior of γ and, consequently, represents an element. We shall find that the point i is a singularity of this element. In fact, if we assume the contrary, then we must have a neighbourhood U of the point i and a function $\varphi(z)$, analytic in this neighbourhood and coinciding with $f(z)$ in that part of the neighbourhood U which lies in the interior of γ (this part will be denoted by d). Then, at the point i there must exist the finite limit

$$\varphi(i) = \lim_{\substack{z\to i \\ z\in d}} \varphi(z) = \lim_{\substack{z\to i \\ z\in d}} f(z) = \lim_{\substack{z\to i \\ z\in d}} \frac{1}{1+z^2},$$

which is, however, impossible.

Thus, the singularity of an element of $f(z)$ lies on γ and, consequently, the radius of convergence R of the Taylor expansion of $f(z)$ in powers of $z - z_0$ coincides with the radius ρ of the circle γ, i.e. with the distance of z_0 from the nearer of the two points $\pm i$.

The Taylor expansion in this case is very simple to obtain if we expand $\frac{1}{1+z^2}$ into partial fractions and then make use of the geometric

series. We thus obtain

$$\frac{1}{1+z^2} = -\frac{1}{2i}\left(\frac{1}{i-z}+\frac{1}{i+z}\right) =$$

$$= -\frac{1}{2i}\left(\frac{1}{i-z_0}\,\frac{1}{1-\dfrac{z-z_0}{i-z_0}} + \frac{1}{i+z_0}\,\frac{1}{1+\dfrac{z-z_0}{i+z_0}}\right).$$

Since $\left|\frac{z-z_0}{i-z_0}\right| < 1$ and $\left|\frac{z-z_0}{i+z_0}\right| < 1$ (the point z lies in the interior of γ while the points $\pm i$ lie either both on γ or one on γ and the other in the exterior of γ), each of the fractions $\dfrac{1}{1-\dfrac{z-z_0}{i-z_0}}$ and $\dfrac{1}{1+\dfrac{z-z_0}{i+z_0}}$ can be represented by a geometric series with common ratios $\frac{z-z_0}{i-z_0}$ and $-\frac{z-z_0}{i+z_0}$, respectively. Thus, we obtain

$$\frac{1}{1+z^2} = -\frac{1}{2i}\left[\frac{1}{i-z_0}\sum_0^\infty \left(\frac{z-z_0}{i-z_0}\right)^n + \frac{1}{i+z_0}\sum_0^\infty (-1)^n\left(\frac{z-z_0}{i+z_0}\right)^n\right] =$$

$$= \frac{1}{2i}\sum_0^\infty (-1)^{n+1}\left[(z_0+i)^{-n-1} - (z_0-i)^{-n-1}\right](z-z_0)^n.$$

This is an expansion whose radius of convergence R has already been determined. As a particular case, for $z_0 = 1$, we obtain $R = \rho = \sqrt{2}$, and the expansion takes the form

$$\frac{1}{1+z^2} = \sum_0^\infty (-1)^n 2^{-\frac{n+1}{2}}\left[\sin(n+1)\frac{\pi}{4}\right](z-1)^n =$$

$$= \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 - \frac{1}{8}(z-1)^4 + \ldots .$$

Example 2. Let $F(z) = \dfrac{1}{\operatorname{Ln} z}$. We now examine the domain G whose boundary is the nonnegative part of the real axis $\Lambda: x \geqslant 0, y = 0$. In this domain the many-valued function $F(z)$ can be split into one-valued analytic branches; of these branches we examine the one

$$f(z) = \frac{1}{\operatorname{Ln}_1 z} = \frac{1}{\ln|z| + i\operatorname{Arg}_1 z},$$

where $\operatorname{Arg}_1 z$ satisfies the inequalities

$$0 < \operatorname{Arg}_1 z < 2\pi.$$

It follows from the formula defining $F(z)$ [or $f(z)$] that the singularities of the elements of the function $F(z)$ can coincide either with $z = 0$ [the branch point of Ln z and $F(z)$] or with the point $z = 1$ at which one of the values of the logarithm becomes zero. We put, for the sake of definiteness, $z_0 = 1 + i$. Then, of the two points 0 and 1, the point 1 will, obviously, be the point nearer to z_0. Describing a circle γ with z_0 as its centre and of radius ρ equal to unity (Fig. 59), we find that $f(z)$, in the interior of γ, defines certain element $\varphi(z)$. As the point z (lying in the interior of γ) tends to the point 1 (lying on γ), $\ln |z|$ tends to zero and also $\mathrm{Arg}_1 z$ tends to zero; hence, the values of the element $\varphi(z)$ tend to ∞. From this, arguing as in the previous example, we conclude that the point $z = 1$ is a singularity of the element in question. Hence, the radius of convergence R of the Taylor expansion of $\varphi(z)$ [the expansion in powers of $z - (1+i)$] is unity.

We also examine the point $z_0' = 1 - i$, which lies in the lower half-plane. For this, 1 will again be the nearer of the two points 0 and 1. Describing a circle γ' with z_0' as its centre and of radius equal to 1, we shall get in its interior a definite element of the function $f(z)$; this element we shall denote by $\psi(z)$. We shall now show that all the points of the circle γ' are regular points of this element. To prove this, we shall take the circle K: $|z - (1-i)| < \sqrt{2}$, the boundary of which contains the point 0. In the interior of the circle K, the function Ln z can be split into one-valued analytic branches, one of which coincides with $\mathrm{Ln}_1 z$ in the interior of the circle γ': (it is sufficient to choose a one-valued branch of Ln z, which has a value coinciding with $\mathrm{Ln}_1 z$ at the point z_0'). We denote this one-valued branch of Ln z by $\mathrm{Ln}_2 z = \ln |z| + i \mathrm{Arg}_2 z$. Since the value of $\mathrm{Arg}_2 z$ coincides with the value of $\mathrm{Arg}_1 z$, equal to $\frac{7\pi}{4}$, at the point z_0' and the values of $\mathrm{Arg}_2 z$ in the interior of the circle K deviate from the value at the centre of this circle by less than $\frac{\pi}{2}$ in absolute magnitude,

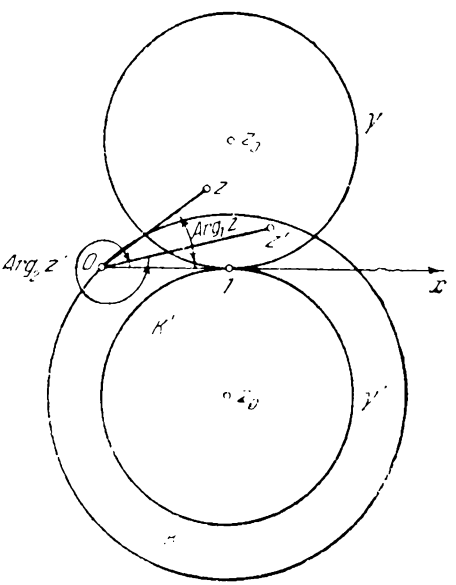


Fig. 59.

hence $\operatorname{Arg}_2 z$, at all points of the circle K, will satisfy the inequalities

$$\frac{5\pi}{4} < \operatorname{Arg}_2 z < \frac{9\pi}{4}.$$

Thus, it follows that $\operatorname{Ln}_2 z$ does not vanish in the circle K. So the function $\frac{1}{\operatorname{Ln}_2 z}$ is analytic in this circle. But it coincides with $\frac{1}{\operatorname{Ln}_1 z}$ in the circle K': $|z-(1-i)|<1$, i.e. it coincides with the element $\psi(z)$ of the function $f(z)$. Hence, it follows that every point of the circle γ' is a regular point of $\psi(z)$ (in fact, for any point $\zeta \in \gamma'$, there exist a neighbourhood U_ζ and a function $\frac{1}{\operatorname{Ln}_2 z}$ analytic in it and coinciding with $\psi(z)$ in the part common to U_ζ and K'). In particular, the point 1 will also be a regular point of $\psi(z)$. From this example we can see that one and the same point 1 is a singularity of one element of the analytic function $f(z)$ [of $\varphi(z)$] and a regular point of another element of this very function [of $\psi(z)$].

All points of the circle γ' are regular points of $\psi(z)$; hence, it follows that the radius of convergence of the Taylor expansion of this element in powers of $z-(1-i)$ is greater than the radius of the circle γ', i.e. greater than unity. But, in the circle K', $\psi(z) = \operatorname{Ln}_2 z$; hence, the Taylor expansions of the functions $\psi(z)$ and $\operatorname{Ln}_2 z$ coincide. Since $\operatorname{Ln}_2 z$ is a function which is analytic in the circle $|z-(1-i)|<\sqrt{2}$, the radius of convergence R' of this expansion cannot be less than $\sqrt{2}$. In order to show that it is exactly equal to $\sqrt{2}$, it is sufficient to prove that at least one of the points of the circle $\Gamma: |z-(1-i)| = \sqrt{2}$ is a singularity of $\frac{1}{\operatorname{Ln}_2 z}$. The origin of coordinates is, indeed, such a point. In fact, $\left(\frac{1}{\operatorname{Ln}_2 z}\right)' = -\frac{1}{z(\operatorname{Ln}_2 z)^2} \to \infty$ as $z \to 0$, and we are confined to the interior of Γ. Hence, it also follows that there does not exist any function $\chi(z)$ which is analytic in the neighbourhood U of the origin of coordinates and which coincides with $\frac{1}{\operatorname{Ln}_2 z}$ at the points common to U and K (if we assume the existence of such a function, it is necessary to assume also the existence of the finite limit

$$\chi'(0) = \lim_{z\to 0} \chi'(z) = \lim_{z\to 0} \left(\frac{1}{\operatorname{Ln}_2 z}\right)',$$

which, as we have already noted, is impossible).

We propose to the reader to prove that in the given example, for any point z_0 of the domain G lying in the upper half-plane, the radius of

convergence of the Taylor expansion of the function $f(z)$ in powers of $z - z_0$ is equal to the distance from z_0 to the nearer of the two points 0 and 1, while for the point z_0' lying in the lower half-plane, the distance from z_0' to 1 does not play any role in the determination of the radius of convergence of the corresponding series; moreover, this radius here always coincides with the distance from z_0' to the point 0.

9. ISOLATED SINGULARITIES OF A MULTIPLE CHARACTER

We shall apply the results of the previous articles to the study of functions in the neighbourhood of isolated singularities, generally of a multiple character (i.e. branch points). Let z_0 be a point in the plane (assumed, for definiteness, to be finite) and let G_j ($j=0, \pm 1, \pm 2, \ldots$) denote alternately the upper and lower semi-circles of a certain radius $R \leqslant \infty$ with their centre at this point. In fact, denoting the polar coordinates, with origin at z_0, by φ and r, we will define G_j by the inequalities $(j-1)\pi < \varphi < j\pi$ $(0<r<R)$ [obviously, we take the upper semi-circle when j is odd and the lower one when j is even]. Let $\{G_j, f_j(z)\}$ be elements such that $f_j(z)$, for any $j = 0, \pm 1, \pm 2, \ldots$, can be continued through the radius $\varphi = j\pi$ $(0 < r < R)$ into the domain G_{j+1}, f_{j+1} being the result of this continuation. Then, the set of elements $\{G_j, f_j(z)\}$ defines a function $f(z)$ analytic in the neighbourhood $0 < |z - z_0| < R$ of the point z_0. If the element $\{G_3, f_3(z)\}$ coincides with the element $\{G_1, f_1(z)\}$, then, as can be easily seen, any element $\{G_{j+2}, f_{j+2}\}$, coincides with the element $\{G_j, f_j(z)\}$ $(j=0, \pm 1, \pm 2, \ldots)$ and the function $f(z)$ is found to be one-valued. In the upper semi-circle G_1, it coincides with $f_1(z)$ whereas in the lower semi-circle G_2 with $f_2(z)$; in this case, the continuation of $f_1(z)$ through the radius $\varphi = \pi$ $(0<r<R)$ into the lower semi-circle gives $f_2(z)$, while the continuation of $f_2(z)$ through the radius $\varphi = 2\pi$ $(0<r<R)$ into the upper semi-circle gives $f_1(z)$. The point $z = z_0$ can either be a regular point of $f(z)$ or a singularity (isolated singularity of a simple character). This case has already been discussed in detail (Chapter 7). We now assume that the element $\{G_3, f_3(z)\}$ is different from the element $\{G_1, f_1(z)\}$, that is, the function $f_3(z)$ is different from the function $f_1(z)$ in the upper semi-circle. Now we shall further examine the elements $\{G_5, f_5(z)\}, \ldots, \{G_{2k+1}, f_{2k+1}(z)\}$ $(k>1)$ comparing each of them with $\{G_1, f_1(z)\}$. There are only two possibilities here. Either we find an element in the infinite set of elements $\{G_{2k+1}, f_{2k+1}\}$ that coincides with $\{G_1, f_1(z)\}$ or we do not find such an element. We examine the first possibility. Let $\{G_{2k_0+1}, f_{2k_0+1}(z)\}$ $(k_0>1)$ be an element with the lowest ordinal number which coincides with $\{G_1, f_1(z)\}$. We shall now show that

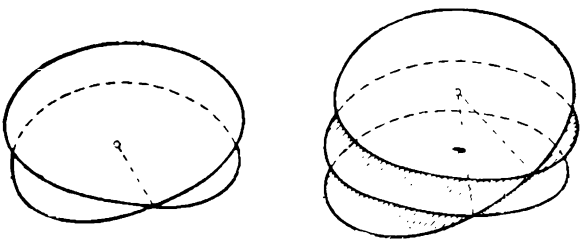
the elements $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$ and $\{G_j, f_j(z)\}$ will coincide with each other for any integral value of j. Evidently, it is sufficient to show for this purpose that the coincidence of the elements $\{G_{j_0\pm1+2k_0}, f_{j_0\pm1+2k_0}(z)\}$ and $\{G_{j_0\pm1}, G_{j_0\pm1}(z)\}$ follows from the coincidence of the elements $\{G_{j_0+2k_0}, f_{j_0+2k_0}(z)\}$ and $(G_{j_0}, f_{j_0}(z)\}$ for any integral value of j_0. But, if the analytic functions $f_{j_0}(z)$ and $f_{j_0+2k_0}(z)$ coincide in the semi-circle $G_{j_0} = G_{j_0+2k_0}$, then the results of their analytic continuation through one and the same radius into another semi-circle $G_{j_0+1} = G_{j_0+1+2k_0}$ will also coincide (by virtue of the uniqueness theorem). A similar assumption that the functions $f_{j_0-1}(z)$ and $f_{j_0-1+2k_0}$ do not coincide in the semi-circle $G_{j_0-1} = G_{j_0-1+2k_0}$ would give us the result that $f_{j_0}(z)$ and $f_{j_0+2k_0}(z)$ (the results of the continuation of the above-mentioned functions through one and the same radius into the semi-circle $G_{j_0} = G_{j_0+2k_0}$) would not coincide. It follows from these arguments that the coincidence of $\{G_1, f_1(z)\}$ with $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$ necessitates the coincidence of $\{G_j, f_j(z)\}$ with $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$ for all values of j. Consequently, in the infinite set of elements $\{G_n, f_n(z)\}$, there can exist only a finite number of different elements $\{G_1, f_1(z)\}, \ldots \{G_{2k_0}, f_{2k_0}(z)\}$. It can be easily seen that all of them are really different. Let us assume the contrary and let the elements $\{G_p, f_p(z)\}$ and $\{G_q, f_q(z)$, where $1 \leqslant p < q \leqslant 2k_0$, coincide with each other. Using the result established above, we find that the elements $\{G_j, f_j(z)$ and $\{G_{j+q-p}, f_{j+q-p}(z)\}$ will also coincide for any integral value of j. In particular, the elements $\{G_1, f_1(z)\}$ and $\{G_{1+q-p}, f_{1+q-p}(z)\}$, where $0 < q - p < 2k_0$, must coincide with each other. But this result contradicts the assumption that $2k_0$ is the lowest of the natural numbers n for which the element $\{G_{1+n}, f_{1+n}(z)\}$ coincides with $\{G_1, f_1(z)\}$.

Thus, the k_0 functions $f_1(z), f_3(z), \ldots, f_{2k_0-1}(z)$ are different from one another in the semi-circle G_1 just as the k_0 functions $f_2(z), \ldots,$ $, \ldots, f_{2k_0}(z)$ are different from one another in the semi-circle G_2. Hence, it follows that the analytic function $f(z)$, defined by the elements $\{G_j, f_j(z)\}$ in the domain $0 < |z - z_0| < R$, is a many-valued, i.e. a k_0-valued function. Now, taking a one-valued branch $f_j(z)$ of this function in one of the semi-circles G_j and continuing it analytically into the semi-circles $G_{j+1}, G_{j+2}, \ldots$ through the corresponding radii, we shall, on return to the original semi-circle, always meet a new branch $f_{j+2}(z)$, $f_{j+4}(z), \ldots,$ until we arrive at $f_{j+2k_0}(z)$ coinciding with $f_j(z)$. We can describe this situation as follows.

A revolution around the point z_0 during analytic continuation in the anti-clockwise direction results in the substitution of the element $\{G_j, f_j(z)\}$ by a different one, viz. $\{G_{j+2}, f_{j+2}(z)\}$, and it is only after k_0

revolutions about this point that we arrive at the element $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$ coinciding with the original element. The point z_0 in this case is called a *branch point of a finite order* $(=k_0-1)$ *of the function* $f(z)$. It can easily be seen that the Riemann surface $f(z)$, in the neighbourhood of the point z_0, will have in this case the form of a k_0-*sheeted circle* (Fig. 60 *a* and *b*, where the cases for $k_0=2$ and $k_0=3$ have been diagrammatically represented).

We now derive an analytic expression for $f(z)$ in the neighbourhood of a branch point of order (k_0-1). To do this, we subject the k_0-



a) *b*)

Fig. 60.

sheeted circle to the transformation $t=\sqrt[k_0]{z-z_0}$. Then, the semicircles $G_1, G_2, \ldots, G_{2k_0}$, composing the Riemann surface, will be transformed into the sectors $g_1, g_2, \ldots, g_{2k_0}$ of a circle of radius $\sqrt[k_0]{R}$, with its centre at the origin of coordinates, the sector g_j being defined by the inequalities $(j-1)\frac{\pi}{k_0}<\alpha<j\frac{\pi}{k_0}$, $0<\rho<\sqrt[k_0]{R}$ (α and ρ being polar coordinates with origin at the point $t=0$). The function $f_j(z)$ is transformed into the function $f_j^*(t)=f_j(z_0+t^{k_0})$, while $f_j^*(t)$ would admit of an analytic continuation through the radius $\alpha=j\frac{\pi}{k_0}$, $0<\rho<\sqrt[k_0]{R}$ into the neighbouring sector g_{j+1}, the result of such a continuation being $f_{j+1}^*(t)$. The elements $\{g_j, f_j^*(t)\}$ $(j=1, 2, \ldots, 2k_0)$ define, in the domain $0<|t|<\sqrt[k_0]{R}$, a one-valued analytic function $f^*(t)$ which will, consequently, be represented in this domain by its Laurent expansion $f^*(t)=\sum_{-\infty}^{+\infty}A_nt^n$. For the coefficients $A_{-1}, A_{-2}, \ldots,$ $, \ldots, A_{-n}, \ldots,$ we have one of the three possibilities : (*a*) an inifinite set of these coefficients is nonvanishing, (*b*) only a finite number $(\geqslant 1)$ of these is nonvanishing, and (*c*) all these coefficients vanish. The case (*a*) obtains when $\lim_{t\to 0} f^*(t)$ does not exist at all. The case (*b*) holds when

$\lim_{t\to 0} f^*(t) = \infty$ and the case (*c*) when $\lim_{t\to 0} f^*(t)$ exists and is different from ∞. Now, returning to the variable z, we conclude that $f(z)$ has the expansion

$$f(z) = \sum_{-\infty}^{+\infty} A_n (z-z_0)^{\frac{n}{k_0}}$$

in terms of integral powers of $(z-z_0)^{\frac{1}{k_0}}$. Here, coefficients of terms with negative indices satisfy the conditions (*a*), (*b*) or (*c*) depending, respectively, upon whether $\lim_{z\to z_0} f(z)$ does not exist at all, actually exists and is equal to infinity or, finally, if it exists and is finite.

In the last two cases (i.e. when there exists a finite or an infinite limit $\lim_{z\to z_0} f(z)$, the point z_0 is called an *algebraic branch point.*

If, however, the limit $f(z)$ does not exist, the branch point z_0 of order $(k_0 - 1)$ is called a *transcendental branch point of a finite order* $(=k_0-1)$. It can be easily seen that the functions $\sqrt[k_0]{z-z_0}$, $\frac{1}{\sqrt[k_0]{z-z_0}}$, $e^{\sqrt[k_0]{z-z_0}}$, and $\sin \sqrt[k_0]{z-z_0}$ have, for $z = z_0$, an algebraic branch point of order $k_0 - 1$, while the functions $e^{\frac{1}{\sqrt[k_0]{z-z_0}}}$ and $\sin \frac{1}{\sqrt[k_0]{z-z_0}}$ have, for $z = z_0$, a transcendental branch point of the same order: $k_0 - 1$.

Now, returning to the general case, we assume that none of the elements $\{G_j, f_j(z)\}$ for $j > 1$ coincides with $\{G_1, f_1(z)\}$. Then, it follows from the above that none of the two elements $\{G_p, f_p(z)\}$ and $(G_q, f_q(z)\}$, for $p \neq q$, can coincide. In this case, the analytic function $f(z)$, defined in the domain $0 < |z - z_0| < R$ by the totality of elements $\{G_n, f_n(z)\}$, is found to be infinite-valued, while the analytic continuation of any element $\{G_j, f_j(z)\}$, after any number of traversals round the point z_0, leads to an element different from the original. The point z_0 in this case is called a *branch point of an infinite order* or a *logarithmic branch point.* Then, the Riemann surface $f(z)$, in the neighbourhood of the point z_0, has the form of an infinite-sheeted circle. The functions $\operatorname{Ln}(z-z_0)$ and $(z-z_0)^\alpha$ (α is not a real rational number) can be cited as examples of functions with logarithmic branch points at the point z_0. The function $\operatorname{Ln}\frac{z-a}{z-b}$ has two logarithmic branch points: $z = a$ and $z = b$. The

function $\operatorname{Arctan} z = \frac{1}{2i} \operatorname{Ln} \frac{i-z}{i+z}$ has two logarithmic branch points: $z = \pm i$.

All that has been said in this article can also be applied to the point $z = \infty$ if we consider, for example, its neighbourhood, $R < |z| < +\infty$, divided into two parts lying in the upper and the lower half-planes, respectively. It is sufficient to use a transformation of the form $\zeta = \frac{1}{z}$ so as to reduce this case to that of the finite point $\zeta = 0$.

We note that the term algebraic branch point is justified by the following theorem.

Theorem. *Among analytic functions it is only the algebraic functions $F(z)$, i.e. the functions satisfying an equation of the type* $\sum_{k=0}^{N} P_k(z)\,[F(z)]^k = 0$ [$P_k(z)$ *are polynomials and* $P_N(z) \not\equiv 0$, $N \geqslant 1$], *that possess the property that their singularities in the extended plane can only be poles and algebraic branch points.*[1]

Example 1. The singularities of the algebraic function $\sqrt[3]{z^2-5z+6}$ are the points $z = 2$, $z = 3$ and $z = \infty$. Each of these is an algebraic branch point of the second order.

Example 2. The function $\sqrt[3]{1+z^3}$ has three algebraic branch points of the second order at $z = -1$, $e^{\frac{\pi i}{3}}$, $e^{-\frac{\pi i}{3}}$ and a simple pole at $z=\infty$ [to be more precise, each of the three branches of the function $\sqrt[3]{1+z^3} = z\sqrt[3]{1+z^{-3}} = ze^{\frac{2\pi ik}{3}}\left(1+\frac{1}{3}z^{-3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}z^{-6}+\ldots\right)$, where $k = 0, 1, 2$, one-valued in the neighbourhood $|z| > 1$ of the point $z = \infty$, has a simple pole at the point $z = \infty$].

Example 3. The function $e^{\sqrt{z}}$ has an algebraic branch point of the second order at the point $z = 0$ and a transcendental branch point of the same order at $z = \infty$.

Example 4. The function $\sqrt{\cos z}$ has an algebraic branch point of the first order at $z = \frac{\pi}{2} + k\pi$ ($k = 0, \pm 1, \pm 2 \ldots$) and a transcendental branch point of the same order at $z = \infty$.

[1] See, in this connection, A. I. Markushevich, loc. cit., Chapter 8, Art. 6.

Example 5. The functions $\frac{\sin\sqrt{z}}{\sqrt{z}}$, $\cosh\sqrt{z}$, $e^{\sqrt[3]{z}}+e^{\varepsilon\sqrt[3]{z}}+e^{\varepsilon^2\sqrt[3]{z}}$, $\varepsilon=e^{\frac{2\pi i}{3}}$ being a cube root of 1, are entire functions (with one essential singularity at infinity), because after one traversal around any possible branch point (either the origin of coordinates or the point at infinity) they would resume their original values.

10

MAPPINGS BY ANALYTIC FUNCTIONS. THE CONCEPT OF ELLIPTIC FUNCTIONS. THE CHRISTOFFEL-SCHWARTZ FORMULA

1. MAPPING OF A DOMAIN BY AN ANALYTIC FUNCTION

Let $w = f(z) \not\equiv \text{const}$ be a function, one-valued and analytic in the domain G. We shall show that the set D of all the values taken by it in G is also a domain. For this, we have to show that any point $w_0 = f(z_0) \in D$, along with its certain neighbourhood, belongs to D and that any two points in D can be joined by a continuous curve lying wholly in D. We construct a closed circle $C_0 : |z-z_0| \leqslant \rho$, such that it is contained in the domain G and the value w_0 is not taken by the function at any point other than z_0 of the circle. This last requirement can be met if we take ρ sufficiently small. Assuming the contrary, we would have, in any neighbourhood of z_0, points at which $f(z)$ takes up one and the same value w_0; but this is impossible by the uniqueness theorem [$f(z) \not\equiv w_0$]. We denote $\min\limits_{|z-z_0|=\rho} |f(z) - w_0|$ by μ. Obviously, $\mu > 0$, by virtue of the fact that $f(z)$ is continuous and does not become w_0 on the circle $|z-z_0| = \rho$. We shall show that the circle $K_0 : |w-w_0| < \mu$ belongs to the set D. In fact, the equation $f(z) - w_0 = 0$ has at least one root in the interior of C_0 and that root is $z = z_0$ (this point may be counted as many times as its multiplicity). Let $w' \in K_0$ ($w' \neq w_0$). Since $|w_0-w'| < \mu$ and $|f(z) - w_0| \geqslant \mu$ on the circle $|z-z_0| = \rho$, we find by the Rouche' theorem that the equation $[f(z)-w_0] + (w_0-w') = f(z) - w' = 0$ has in the interior of this circle as many roots as the equation $f(z) - w_0 = 0$ has, i.e. it has at least one root. We denote one of the roots by z' and thus obtain $f(z') = w'$, i.e. any point w' of the circle K_0 is a value of the function $f(z)$ taken up in the interior of the circle C_0. Consequently, K_0 is contained in D.

Let $w_1 = f(z_1)$ and $w_2 = f(z_2)$ be any two points of the set D. Since z_1 and z_2 belong to G, there exists a continuous curve $\gamma : z = \lambda(t)$,

$\alpha \leqslant t \leqslant \beta$, connecting z_1 with z_2 in the interior of G [$\lambda(\alpha)=z_1$, $\lambda(\beta)=z_2$]. Evidently, $w = f(z) = f[\lambda(t)]$, $\alpha \leqslant t \leqslant \beta$, is also a continuous curve contained in the set D and connecting w_1 with w_2 [$(w_1=f(z_1) = f[\lambda(\alpha)]$, $w_2 = f(z_2) = f[\lambda(\beta)])$].

Thus, we have proved that *the set D of the values taken up by the one-valued analytic function* $w = f(z) \not\equiv$ const, *in some domain G, is also a domain. In other words, an analytic function* ($\not\equiv$ const) *always transforms a domain into a domain.*

The theorem proved above also remains valid when the domain G of the extended plane contains the point ∞ and $f(z)$ possesses poles in the domain G. To show this, it is sufficient to carry out an auxiliary bilinear mapping. That is, if $z_0 = \infty$, we shall carry out the mapping $z' = 1/z$ which transforms the neighbourhood of the point at infinity into the neighbourhood of the origin of coordinates. If $w_0 = f(z_0) = \infty$ [z_0, either a finite point or the point at infinity, being a pole of the function $f(z)$], we carry out the auxiliary mapping $w' = 1/f(z)$ which transforms the function $f(z)$ with a pole at z_0 into the function $1/f(z)$ with a zero at the same point. It is evident that these transformations reduce the problem to the case already considered in this article, in which the values of the independent variable and the function are finite. Similar remarks can also be made in regard to the succeeding article.

2. THE PRINCIPLE OF THE MAXIMUM MODULUS AND THE SCHWARTZ LEMMA

The principle of the maximum modulus follows from the theorem of Art. 1. *Let* $w = f(z) \not\equiv$ const *be a function which is one-valued and analytic at all points of the domain G of the extended plane. Then, the modulus* $|f(z)|$ *cannot have its maximum at any point* $z_0 \in G$. In other words, if we know that the modulus of the function $f(z)$ which is one-valued and analytic in the domain G has a maximum at a certain point of this domain, then $f(z) \equiv$ const.

Proof. In the first formulation, the proof follows from the fact that the point $w_0 = f(z_0)$ belongs to the set of values of the function $f(z)$ along with a certain neighbourhood K_0: $|w-w_0| < \mu$ (we maintain the symbols of Art. 1). We choose a point w' in this neighbourhood such that $|w'| > |w_0|$. By what has been proved in Art. 1, $w' = f(z')$, where z' lies in the interior of the neighbourhood C_0: $|z-z_0| \leqslant \rho$ of the point z_0. Thus, in the neighbourhood of the point z_0 (this neighbourhood can be made as small as we like by choosing ρ sufficiently small) we can always find a point z' at which $|f(z')| > |f(z_0)|$. This also means that the modulus $|f(z)|$ cannot have its maximum at the point z_0. The reader can also

prove exactly in the same way that when $f(z)$ does not vanish in the domain G, its modulus cannot have a minimum at any point of the domain. Now, we note an important corollary to the principle of the maximum modulus.

If $f(z)$ is continuous in the closed domain $\overline{G}$ and also analytic within this domain, then $\sup |f(z)|$ *is attained on the boundary Γ of the domain.* In fact, the modulus $|f(z)|$, being continuous on the closed point set $\overline{G}$, must attain its upper bound, at least at one of the points of this set. But, if $f(z) \not\equiv$ const, there cannot be such a point in the interior of G. Consequently, it can and must be some point on the boundary. If, however, $f(z) \equiv$ const, then $\sup |f(z)|$ is attained at every point of $\overline{G}$ and also at a boundary point. Thus, the statement is proved for all cases. Noting that in this proposition we can write $\max\limits_{z \in \Gamma} |f(z)|$ instead of $\sup\limits_{z \in \Gamma} |f(z)|$, we arrive at the result that at every point of the domain G the inequality

$$|f(z)| \leqslant \max_{z \in \Gamma} |f(z)|$$

will be satisfied, and that the sign of equality will hold for any point $z_0 \in G$ if, and only if, $f(z) \equiv$ const.

As an application of the principle of maximum modulus, we shall prove the so-called Schwartz lemma which will be used later.

The Schwartz lemma. *If $f(z)$ is a function which is one-valued and analytic in the unit circle and satisfies the conditions $f(0) = 0$ and $|f(z)| \leqslant 1$ ($|z| < 1$), then it also satisfies the conditions $|f'(0)| \leqslant 1$ and $|f(z)| \leqslant |z|$ ($|z| < 1$). Here, the equality $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ (even at an isolated point z_0, $z_0 \neq 0$, $|z_0| < 1$) can be true only when $f(z)$ is a linear function of the form $f(z) = e^{i\alpha} z$* (α being a real constant).

Proof. Let $f(z) = c_1 z + c_2 z^2 + \ldots$ ($|z| < 1$); we put $\varphi(z) = f(z)/z = c_1 + c_2 z + \ldots$. It is obvious that $\varphi(z)$ is a function analytic in the unit circle, satisfying the condition $\varphi(0) = c_1 = f'(0)$.

We now examine the value of $\varphi(z)$ at a point z' of the unit circle. If r satisfies the condition $|z'| < r < 1$, then we must have, according to the foregoing,

$$|\varphi(z')| \leqslant \max_{|z| = r} |\varphi(z)|.$$

But, $\max\limits_{|z| = r} |\varphi(z)| = \max\limits_{|z| = r} \left|\dfrac{f(z)}{z}\right| \leqslant \dfrac{1}{r}$, because $|f(z)| \leqslant 1$; hence,

$|\varphi(z')| \leqslant \frac{1}{r}$ or, keeping z' fixed and making r tend to unity,

$$|\varphi(z')| \leqslant 1.$$

Especially, for $z' = 0$, $|\varphi(0)| = |f'(0)| \leqslant 1$ and for $z = z_0 \neq 0$, $|\varphi(z_0)| = \left|\frac{f(z_0)}{z_0}\right| \leqslant 1$, i.e. $|f(z_0)| \leqslant |z_0|$.

The sign of equality in one of these relations indicates that at some point z' of the unit circle the modulus $|\varphi(z)|$ attains its maximum value, which is equal to unity. This is possible only when $\varphi(z) \equiv$ $\equiv$ const $= e^{i\alpha}$ (because $|\varphi(z)| = 1$), i.e. $f(z) = e^{i\alpha}z$.

We now apply the principle of the maximum modulus to harmonic functions also. We shall prove that *a function $u(x, y) \not\equiv$ const, which is one-valued and harmonic in the domain G, can have neither a maximum nor a minimum at any point of the domain.* Now, let (x_0, y_0) be a point of the domain G and U_0 be its neighbourhood contained in G. We construct the function $v(x, y)$ harmonic in G and conjugate to $u(x, y)$ (Art. 12, Chapter 2). Then, $u(x, y) + iv(x, y) = f(z)$ will be a function which is one-valued and analytic in U_0. Then, the functions $F_1(z) = e^{f(z)}$ and $F_2(z) = e^{-f(z)}$ will also be one-valued and analytic in the same neighbourhood. Since these are not constants, the principle of the maximum modulus is applicable to each one of them. Hence, the values of $|F_1(z_0)| = e^{u(x_0, y_0)}$ and $|F_2(z_0)| = e^{-u(x_0, y_0)}$ cannot be maximum; consequently, the value $u(x_0, y_0)$ of a harmonic function can neither be a maximum nor a minimum.

If $u(x, y)$ is a function continuous in a closed bounded domain G and also harmonic in this domain, then, from what has been proved above, it follows that *its greatest and least values must be attained only at the boundary points of the domain.* If, in particular, $u(x, y)$ preserves a constant value on the boundary of the domain G, it would follow that $u(x, y) \equiv$ const in this domain. Hence, the two functions $u_1(x, y)$ and $u_2(x, y)$, which are continuous in the closed domain $\overline{G}$, are harmonic in this domain and take up identical values at the boundary points of the domain G, must coincide with each other everywhere in this domain. This means that the so-called *Dirichlet problem* of determining a function, which is continuous in the closed domain $\overline{G}$ and harmonic in this domain, from the values given on the boundary of the domain, can have only one solution.

3. LOCAL CRITERION OF SINGLE-SHEETEDNESS

We now examine in detail the mapping $w = f(z)$ in the closed neighbourhood $C_0: |z - z_0| \leqslant \rho$. We first assume that $f'(z_0) = 0, \ldots,$

$\ldots, f^{(p-1)}(z_0) = 0, f^{(p)}(z_0) \neq 0$ $(p \geqslant 2)$. The derivative $f'(z)$ can vanish not only at the point z_0 but also at other points of C_0. Since z_0 is not a limit point for the set of zeros of $f'(z)$ [otherwise, we would have $f'(z) \equiv 0$ and $f(z) \equiv$ const] hence, we can choose ρ to be so small that $f'(z)$ does not vanish at any point of C_0 different from z_0. Let us assume that such a choice has already been made. Then, preserving the various conditions and notations, we can prove that, for any $w' \neq w_0$, $|w'-w_0| < \mu$, the equation $f(z) - w' = 0$ has in the interior of C_0 as many roots as the equation $f(z) - w_0 = 0$. But the latter has p roots (z_0 is a p-tuple root of the equation). So, $f(z) - w' = 0$ also has p roots in the interior of C_0. None of these roots z' can be a multiple root. Since $z' \neq z_0$ [otherwise, we would have $w' = f(z') = f(z_0) = w_0$, which is impossible] ; hence, $f'(z') \neq 0$.

Thus, the equation $f(z) - w' = 0$ has p different roots, $z_1, z_2, \ldots, z_p$, in the interior of C_0, i.e. there exist p different points in this neighbourhood at which $f(z)$ takes one and the same value w'. It directly follows that the function $f(z)$, whose derivative vanishes at least at one point z_0 of the domain G, cannot be single-sheeted in this domain. In other words, *if $f(z)$ is single-sheeted in the domain G, then its derivative cannot vanish at any point of the domain.*

The condition $f'(z) \neq 0$ $(z \in G)$, though necessary, does not appear to be sufficient for single-sheetedness ; this may be illustrated by considering the function e^z. Its derivative $(e^z)' = e^z$ does not vanish anywhere and e^z takes one and same value at all points given by $z + 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

We, however, show that *for the point z_0, at which $f'(z_0) \neq 0$, we can always construct a closed neighbourhood $C_0 : |z-z_0| \leqslant \rho_0$, such that $f(z)$ will be single-sheeted in it.* Let $f(z) = \sum_0^\infty a_n (z-z_0)^n$ be the expansion of $f(z)$ in some neighbourhood $|z-z_0| < r$ of the point z_0 $[a_1 = f'(z_0) \neq 0]$. Then, the series $f'(z) = \sum_1^\infty n a_n (z-z_0)^{n-1}$ will converge in this neighbourhood ; consequently, for $\rho < r$, the series $\sum_2^\infty n |a_n| \rho^{n-1}$ will also converge and its sum will tend to zero as $\rho \to 0$. We choose ρ_0 $(0 < \rho_0 < r)$ such that the inequality $\sum_2^\infty n |a_n| \rho_0^{n-1} < |a_1|$ is satisfied. Then, for any two points z_1 and z_2 $(z_1 \neq z_2)$, lying on the closed circle $|z-z_0| \leqslant \rho_0$,

we shall have

$$|f(z_1) - f(z_2)| = \left| \sum_0^\infty a_n [(z_1-z_0)^n - (z_2-z_0)^n] \right| =$$

$$= \left| a_1 (z_1-z_2) + \sum_2^\infty a_n [(z_1-z_0)^{n-1} + \ldots + (z_2-z_0)^{n-1}] (z_1-z_2) \right| \geqslant$$

$$\geqslant |z_1-z_2| \left\{ |a_1| - \sum_2^\infty |a_n| [|z_1-z_0|^{n-1} + \ldots + |z_2-z_0|^{n-1}] \right\} \geqslant$$

$$\geqslant |z_1-z_2| \left\{ |a_1| - \sum_2^\infty n |a_n| \rho_0^{n-1} \right\} > 0, \text{ i.e. } f(z_1) \neq f(z_2).$$

This proves that the function $f(z)$ is single-sheeted in the closed circle $|z-z_0| \leqslant \rho_0$.

4. INVERSION OF AN ANALYTIC FUNCTION

Let $w = f(z)$ be a one-valued analytic function which is single-sheeted in the domain G, not containing the point at infinity. As proved in Art. 1, this function maps G onto a certain domain D. We shall now show that the inverse function $z = \varphi(w)$ is also one-valued, analytic and single-sheeted in D. The one-valuedness of $\varphi(w)$ follows immediately from the single-sheetedness of $f(z)$. Now, if $z_1 \neq z_2$ are two values of $\varphi(w)$ at the point $w_0 \in D$, we must have $f(z_1) = f(z_2) = w_0$; this, however, contradicts the assumption of single-sheetedness of $f(z)$. Similarly, the single-sheetedness of $\varphi(w)$ follows from the one-valuedness of $f(z)$. If we assume $\varphi(w_1) = \varphi(w_2) = z_0$, where $w_1 \neq w_2$, we must have $f(z_0) = w_1$ and $f(z_0) = w_2$, and this contradicts the assumption of one-valuedness of $f(z)$. We shall now prove the continuity of $\varphi(w)$. Let $w_0 \in D$ and $\varphi(w_0) = z_0$. We now construct for z_0 a closed neighbourhood C_0: $|z-z_0| \leqslant \rho$, as in Art. 1. Then, every point w' belonging to the circle K_0: $|w-w_0| < \mu$, where $\mu = \min_{|z-z_0| = \rho} |f(z) - w_0| > 0$, will be a value of $f(z)$ at some point z' lying in the interior of C_0; in other words, $z' = \varphi(w')$ would lie within C_0 if $w' \in K_0$. For any arbitrary positive ε we choose $\rho < \varepsilon$; then, for any $\mu = \mu(\rho)$ corresponding to it, we will have, if $|w'-w_0| < \mu$,

$$|\varphi(w') - \varphi(w_0)| < \rho < \varepsilon.$$

This proves the continuity of $\varphi(w)$ in the domain D. Now it remains to prove the differentiability of $\varphi(w)$ at any point $w_0 \in D$. Let $\varphi(w_0) = z_0$. Then, for $w \neq w_0$, we have $\varphi(w) = z \to \varphi(w_0) = z_0$ as $w \to w_0$ $(z \neq z_0)$; consequently,

$$\lim_{w \to w_0} \frac{\varphi(w) - \varphi(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}$$

$[f'(z_0) \neq 0$, by virtue of the single-sheetedness of $f(z)]$. Thus, the theorem is completely proved.

We now apply this result to a *local inversion* of any arbitrary one-valued (in general, multi-sheeted) analytic function $z = \varphi(w)$, which in some neighbourhood of the point $w_0 = f(z_0)$, $(z_0 \in G)$, is inverse to $w = f(z)$.

First of all, let $f'(z_0) \neq 0$. Then, according to Art. 2, there exists a circle $C_0 : |z - z_0| < \rho_0$ in which the function $f(z)$ is single-sheeted. This circle is mapped by $w = f(z)$, one-to-one and conformally, onto a certain domain D_0 of the w-plane containing the point w_0.

By what has been proved in this article, the inverse function $z = \varphi(w)$ will be one-valued, analytic and single-sheeted in the domain D_0. It

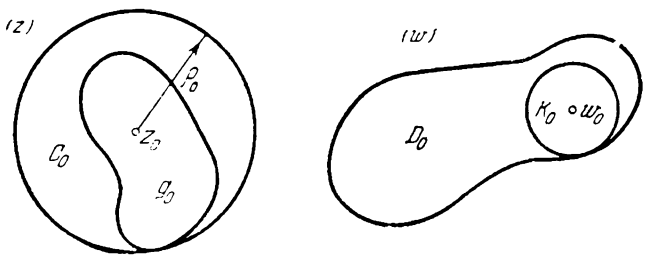


Fig. 61.

maps the circle $K_0 : |w - w_0| < r_0$, r_0 being the distance of the point w_0 from the boundary of the domain D_0, one-to-one and conformally, onto a certain domain g_0 of the z-plane containing the point z_0 (Fig. 61). In the circle K_0, $\varphi(w)$ can be expanded as a power series :

$$z = \varphi(w) = c_0 + c_1(w - w_0) + \ldots + c_n(w - w_0)^n + \ldots \qquad (1)$$

$$(|w - w_0| < r_0).$$

The coefficients c_n of this series can be calculated, for example, by the method of undetermined coefficients [for this, we have to put, instead of $w - w_0 = f(z) - f(z_0)$, the corresponding power series whose coefficients are assumed to be known : $f(z) - f(z_0) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$].

In particular, we obtain

$$c_0 = z_0, \quad c_1 = \frac{1}{a_1}, \ldots .$$

Now, let $f'(z_0) = \ldots = f^{(p-1)}(z_0) = 0$, $f^{(p)}(z_0) \neq 0$ $(p \geqslant 2)$. We express the expansion

$$f(z) = a_0 + a_p (z-z_0)^p + \ldots \left(a_0 = f(z_0) = w_0,\ a_p = \frac{f^{(p)}(z_0)}{p!} \neq 0\right), \quad (2)$$

convergent in the circle $|z-z_0| < \rho$, in the form

$$f(z) = a_0 + a_p (z-z_0)^p [1+\alpha(z)],$$

where $\alpha(z) = \frac{a_{p+1}}{a_p}(z-z_0) + \frac{a_{p+2}}{a_p}(z-z_0)^2 + \ldots$ is a function analytic in the circle $|z-z_0| < \rho$. Let us have ρ_1 $(0<\rho_1<\rho)$ such that $|\alpha(z)| < 1$ in the circle $|z-z_0| < \rho_1$; then, $\beta(z) = [1+\alpha(z)]^{\frac{1}{p}} = 1 + \sum_{n=1}^{\infty} \binom{1/p}{n} [\alpha(z)]^n$ is a function which is one-valued and analytic in this circle and satisfies the condition $\beta(z_0) = 1$. Hence, $\gamma(z) = (z-z_0)\beta(z)$ is also a function which is one-valued and analytic in the circle $|z-z_0| < \rho_1$, for which $\gamma'(z_0) = 1 \neq 0$. Obviously, $f(z)$ can be expressed in terms of $\gamma(z)$ in the form

$$w = f(z) = a_0 + a_p [\gamma(z)]^p. \quad (3)$$

We now examine the transformation

$$\zeta = \gamma(z) = (z-z_0) + b_2 (z-z_0)^2 + \ldots . \quad (4)$$

The results obtained above for the transformation $w = f(z)$, when $f'(z_0) \neq 0$, are applicable to this case as well. Thus, there exists a domain g_0, lying in the circle $|z-z_0| < \rho$ and containing the point z_0, which is mapped by (4), one-to-one and conformally, onto some circle K_0: $|\zeta| < r_1$. The function $z = \delta(\zeta)$, inverse to (4), has in this circle an expansion of the form

$$z = \delta(\zeta) = \zeta + c_2\zeta^2 + \ldots . \quad (5)$$

To obtain the transformation (2) it is sufficient to carry out first the transformation (4) and then subject the circle K_0 to the transformation

$$w = a_0 + a_p\zeta^p \quad \left(a_0 = w_0,\ a_p = \frac{f^{(p)}(z_0)}{p!} \neq 0\right); \quad (6)$$

this, however, is not a one-to-one transformation. It maps p different points of the circle K_0: $\zeta, \zeta e^{i\frac{2\pi}{p}}, \ldots, \zeta e^{i\frac{2\pi}{p}(p-1)}$ $(\zeta \neq 0)$, onto one and the

same point $w = a_0 + a_p\zeta^p$. However, if we represent the image of the circle K_0 by a p-sheeted Riemann surface—a p-sheeted circle with its centre at $a_0 = w_0$, then a one-to-one mapping will take place here. This p-sheeted circle may be constructed as follows. We divide the circle K_0 into $2p$ equal and different sectors S_j: $(j-1)\frac{\pi}{p} < \alpha < j\pi$, $0 < r < r_1$ $(j=1, 2, \ldots, 2p)$ and subject each of these to the transformation (6). Thereupon we obtain $2p$ copies of the upper and the lower semi-circles D_j: $(j-1)\pi < \varphi < j\pi$, $0 < r < r_1^p$ (φ and r being the polar coordinates with origin at the point w_0). Now, what remains is to paste the semi-circles D_j with D_{j+1} $(j=1, 2, \ldots, 2p-1)$ along the common radius $\varphi = j\pi$ $(0<r<r_1^p)$; the last semi-circle D_{2p} is pasted with the first D_1 along the common radius $\varphi = 0$ $(0<r<r_1^p)$. We immediately obtain from the formulae (5) and (6) the inverse $z = \varphi(w)$ of the function $w = f(z)$ in the neighbourhood of the point w_0. In fact, it follows from (6) that

$$\zeta = \left(\frac{1}{a_p}\right)^{\frac{1}{p}} (w-w_0)^{\frac{1}{p}}$$

[here, we can ascribe to the coefficient $\left(\frac{1}{a_p}\right)^{\frac{1}{p}}$ any particular value of the p-th root of $1/a_p$; different values of ζ, corresponding to one and the same w, will be obtained by a traversal around the point w_0]. Substituting this in (5), we get

$$z = \varphi(w) = \sum_1 A_n (w-w_0)^{\frac{n}{p}} \quad \left[A_n = c_n \left(\frac{1}{a_p}\right)^{\frac{n}{p}},\ c_1 = 1\right]. \tag{7}$$

This expansion, which converges for $|w-w_0| < r_1^p$, shows that the function $z = \varphi(w)$ has an algebraic branch point of order $p-1$ at the point w_0. The p-sheeted circle constructed here represents a part of the Riemann surface of the function $\varphi(w)$ in the neighbourhood of the point w_0. It is mapped by (7) onto a single-sheeted domain g_0, lying in the z-plane.

We have shown in this article that *while mapping G by a one-valued analytic function $w = f(z)$, we can, for any point $z_0 \in G$, have a domain g_0, containing the point z_0 and lying in the domain G, which is mapped by $f(z)$ one-to-one onto a single-sheeted* [*if* $f'(z_0) \neq 0$] *or a multi-sheeted* [*if* $f'(z_0) = 0$] *circle with centre at* $w_0 = f(z_0)$. Having chosen the domain g corresponding to any point z, we can paste the corresponding circles $\{K\}$,

joining the two circles K_1 and K_2 if, and only if, they overlap and if their pre-images g_1 and g_2 also overlap. The reader will easily understand that this process is the same as that discussed in Chapter 9 for the construction of the Riemann surface of an analytic function from its elements.

In the given case, we are concerned with the elements of the function $z = \varphi(w)$, defined by an expansion of the form (1) (in a single-sheeted circle) or (7) (in a p-sheeted circle ; $p \geqslant 2$). It is true that the latter case cannot come under the definition of an element (vide Chapter 9), because the sum of the series (7) has a singularity (a branch point) at $w = w_0$. To reduce the form discussed here approximately to the construction given in Chapter 9 it would be sufficient, so to say, to 'unpaste' the p-sheeted circle, i.e. dismember it into semi-circles D_j and take in each semi-circle the corresponding one-valued analytic branch of the function (7): [for example, in the semi-circle $(j-1)\pi < \varphi < j\pi$, $0 < r < r_1^p$, we shall take the one-valued sum of the series $\sum_0^\infty A_n r^{\frac{n}{p}} \left(\cos \frac{n\varphi}{p} + i \sin \frac{n\varphi}{p}\right)$; the values of this branch are given by the image point of that part of the domain g_0 onto which the function (5) maps the circular sector S_j: $(j-1)\frac{\pi}{p} < \alpha < j\frac{\pi}{p}$, $0 < r < r_1$]. However, we would prefer to extend the concept of an element of an analytic function, assuming the p-sheeted circles as domains of the elements and the functions having at the centres of the circles algebraic branch points of $(p-1)$-th order as the functions defined in this domain.

5. THE EXTENSION OF THE CONCEPT OF SINGLE-SHEETEDNESS TO FUNCTIONS HAVING POLES

The case for which z_0 or $w_0 = f(z_0)$ becomes ∞ can be reduced to the case, examined above, as stated at the end of Art. 1. We shall now give the final formulation of certain results.

If $z_0 = \infty$ and the function $f(z)$ is regular at this point, then we have in the neighbourhood of the point at infinity

$$f(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \ldots .$$

It can be easily seen that in this case all the derivatives vanish: $f'(\infty) = f''(\infty) = \ldots = 0$. However, the character of the mapping depends not only on the derivatives but also on the coefficients $A_1, A_2, \ldots$. If $A_1 = A_2 = \ldots = A_{p-1} = 0$ and $A_p \neq 0$ $(p \neq 0)$, then in any neighbourhood of the point at infinity we can find p different

points at which $f(z)$ takes up one and the same value (*cf.* Art. 3). Hence, *in order that the function* $f(z)$ *be single-sheeted in the domain* G *containing the point at infinity, it is necessary that* $A_1 = -\operatorname{Res}_{z=\infty} f(z) \neq 0$. *If only this is known that* $f(z)$ *is regular at the point* $z = \infty$ *and that the coefficient* $A_1 \neq 0$, *then it can be shown that* $f(z)$ *is a single-sheeted function in some neighbourhood of this point* (Art. 3).

We now extend the concept of a single-sheeted function, assuming that a one-valued and analytic function $f(z)$ can have poles in the domain G. From the condition of single-sheetedness [$f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$], it follows that the function can have only one pole. We now show that this pole must necessarily be simple. Let $z_0 \in G$ be a pole of $f(z)$. For the sake of definiteness, we put $z_0 \neq \infty$. Then,

$$f(z) = \frac{A_{-p}}{(z-z_0)^p} + \ldots + \frac{A_{-1}}{z-z_0} + A_0 + \ldots \; (p \geqslant 1,\ A_{-p} \neq 0)$$

in the neighbourhood of the point z_0. If $f(z)$ is single-sheeted in this neighbourhood, then $\frac{1}{f(z)} = \alpha_p (z-z_0)^p + \alpha_{p+1} (z-z_0)^{p+1} + \ldots$ $\ldots \left(\alpha_p = \frac{1}{A_{-p}} \neq 0\right)$ is also single-sheeted in it. By what has been proved in the beginning of Art. 3, this is possible only when $p = 1$, i.e. when the order of the pole of $f(z)$ is unity. In our generalisation of the concept of single-sheetedness, the theorem introduced in the beginning of Art. 4 remains valid also for domains containing the point at infinity, provided that we add in its formulation the condition that the function $z = \varphi(w)$, inverse to the single-sheeted one, can have a simple pole in D.

Finally, we note that *a one-to-one mapping of the domain* G *onto the domain* D, *effected by the single-sheeted function* $w = f(z)$, *is conformal (of the first order) at all points of the domain* G. *If* $z_0 \neq \infty$ *and* $f(z_0) \neq \infty$, then it follows thereby that $f'(z_0) \neq 0$. Let $z_0 \neq \infty$ and $f(z_0) = \infty$; then, $w = f(z) = \frac{A_{-1}}{z-z_0} + A_0 + \ldots (A_{-1} \neq 0)$ and $w' = \frac{1}{w} = \frac{1}{f(z)} = a_1 (z-z_0) + \ldots$, where $a_1 = \frac{1}{A_{-1}} = \left.\frac{dw'}{dz}\right|_{z=z_0} \neq 0$. Since the mapping $w' = \frac{1}{f(z)}$ is conformal at the point z_0, the mapping $w = f(z)$ is also conformal at the same point. The case $z_0 = \infty$ can be reduced to the one examined with the help of the transformation $z' = 1/z$ [here, once again there may be cases for which $f(\infty) \neq \infty$ or $f(\infty) = \infty$. In the first case $w = f(1/z') = A_0 + A_1 z' + A_2 z'^2 + \ldots (A_1 \neq 0)$, and in the second $w = f(1/z') = \frac{A_{-1}}{z'} + A_0 + A_1 z' + \ldots (A_{-1} \neq 0)$].

6. THE CONCEPT OF THE RIEMANN THEOREM. THE UNIQUENESS OF MAPPING

We now examine certain general propositions relating to the theory of conformal mappings. We have seen that any function $w = f(z)$ which is single-sheeted in the domain G [1] maps this domain, one-to-one and conformally (of the first kind), onto a certain domain D.

D. E. Menshov proved the *converse theorem*, viz. *any one-to-one and conformal (of the first order) mapping of a domain G of the extended plane onto another is effected by a certain function which is single-sheeted in the domain G.* Here, in future, we can use the expressions 'conformal mapping of a domain (of the first order)' and 'mapping of a domain by a single-sheeted function' as equivalent.

The theory of conformal mappings yields the following Riemann theorem. We state this without proof : *any simply connected domain G, the boundary of which contains more than one point, can be conformally mapped onto a circle (for example, the unit circle) and that too in an infinite number of different ways.*

The condition imposed here on the domain G is essential. In fact, let the boundary of the domain G consist of only a single point ζ. If $\zeta \neq \infty$, it can be translated into ∞ by the conformal mapping $z' = \frac{1}{z-\zeta}$. We assume this to be accomplished. Then, G coincides with the finite plane. The function $w = f(z)$, mapping G conformally onto the unit circle $|w| < 1$, must be analytic over the whole finite plane, i.e. it must be an entire function and also bounded : $|f(z)| < 1$. By the Liouville theorem (Art. 2, Chapter 6), $f(z) \equiv$ const, i.e. it maps the whole plane onto an isolated point. From this contradiction it follows that a domain, the boundary of which contains only a single point, cannot be conformally mapped onto a circle.

When the conditions of the theorem are satisfied, conformal mapping can be effected in an infinite number of ways. For example, the function $w = f(z)$ which maps the domain G conformally onto the unit circle can be subjected to different complementary conditions. We can, for instance, impose the condition that during the mapping a previously given point z_0 of the domain G goes over to the centre of the circle $[f(z_0)=0]$ and also that the tangents to the curve at this point do not change their sense [i.e. $\arg f'(z_0) = 0$, which means that $f'(z_0)$ is a real positive number]. Now, let $w_1 = f_1(z)$ be any function which maps the domain G

[1] Here, and in what follows, it is implied that this function is one-valued and analytic throughout the domain G, excluding at most a single simple pole.

conformally onto the circle $|w_1| < 1$. In this case, the point z_0 transfers itself into $f_1(z_0)$ and the tangents to the curve at the point z_0 get rotated through an angle equal to $\arg f_1'(z_0) = \alpha$. Now, if $f_1(z_0) \neq 0$ we conformally map the circle onto itself in such a way that the point $f_1(z_0)$ goes over to the centre. This may be effected by the mapping $w_2 = \dfrac{w_1 - f_1(z_0)}{1 - \overline{f_1(z_0)} \cdot w_1}$ (Art. 9, Chapter 3). Obviously, the function $w_2 = f_2(z) = \dfrac{f_1(z) - f_1(z_0)}{1 - \overline{f_1(z_0)}\, f_1(z)}$ maps the domain G conformally onto the circle $|w_2| < 1$ such that the point z_0 goes over to the centre of the circle. Here, $f_2'(z_0) = \dfrac{f_1'(z_0)}{1 - |f_1(z_0)|^2}$ and, consequently, $\arg f_2'(z_0) = \arg f_1'(z_0) = \alpha$.

Now what remains is to put $w = f(z) = e^{-i\alpha} f_2(z)$, so that we could obtain a conformal mapping satisfying the conditions laid down.

We now prove the following *theorem of uniqueness of conformal mapping*. It states that *there can exist only one function $w = f(z)$ which maps the domain G conformally onto the unit circle $K: |w| < 1$, and satisfies the conditions $f(z_0) = 0$, $f'(z_0) > 0$ where z_0 is a fixed point of the domain G.*

Now, let $w_1 = f_1(z)$ and $w_2 = f_2(z)$ be two such functions. It is evident that the function $z = \varphi_2(w_2)$, which is inverse to $w_2 = f_2(z)$, maps the circle $|w_2| < 1$ conformally onto the domain G such that $\varphi_2(0) = z_0$ and $\varphi_2'(0) > 0$. Hence, $w_1 = f_1[\varphi_2(w_2)]$ maps the circle $|w_2| < 1$ onto itself such that $f_1[\varphi_2(0)] = 0$ and $\left.\dfrac{dw_1}{dw_2}\right|_{w_2=0} = \left.\dfrac{dw_1}{dz}\right|_{z=z_0} \cdot \left.\dfrac{dz}{dw_2}\right|_{w_2=0} = f_1'(z_0) : f_2'(z_0) > 0$.

Noting that the Schwartz lemma (Art. 2) is applicable to the function $f_1[\varphi_2(w_2)]$ (because $|f_1 \varphi_2(w_2)| < 1$), we obtain $0 < f_1'(z_0) : f_2'(z_0) \leqslant 1$.

In the above arguments the functions $f_1(z)$ and $f_2(z)$ may change their roles, and then we obtain $0 < f_2'(z_0) : f_1'(z_0) \leqslant 1$. Comparing the two results, we find that $\left.\dfrac{dw_1}{dw_2}\right|_{w_2=0} = f_1'(z_0) : f_2'(z_0) = 1$, which is possible, by virtue of the Schwartz lemma, only under the condition that $w_1 \equiv e^{i\alpha} \cdot w_2$, the factor $e^{i\alpha}$ in this case being necessarily equal to unity $\left(\text{because } \left.\dfrac{dw_1}{dw_2}\right|_{w_2=0} = 1\right)$ Thus, $w_1 = f_1(z) \equiv w_2 = f_2(z)$. This completes the proof.

7. THE CONCEPT OF THE CORRESPONDENCE BETWEEN BOUNDARIES. THE CONVERSE THEOREM

From the Riemann theorem it immediately follows that *any two simply-connected domains G_1 and G_2 in the extended plane, with each of the*

boundaries Γ_1 *and* Γ_2 *containing more than one point, can always be conformally mapped onto each other.* Now, if $w = f_1(z_1)$ maps the domain G onto the circle $|w| < 1$ and $w = f_2(z_2)$ maps the domain G_2 onto the same circle, then $z_2 = \varphi_2 f_1(z_1)$, where $z_2 = \varphi_2(w)$, a function inverse to $w = f_2(z_2)$, maps G_1 onto G_2. We now assume that the boundaries Γ_2 and Γ_1 of the domains G_2 and G_1 are Jordan curves (in the generalised sense, i.e. passing through the point at infinity) and that $z_2 = F(z_1)$ be some single-sheeted function which maps G_1 conformally onto G_2. This function is defined, of course, only at the interior points of the domain G_1. We can, however, prove (we do not give the proof here) that for any point $\zeta_1 \in \Gamma_1$ there exists a definite limit of the function $F(z_1)$ as z_1, remaining all the while in the interior of G_1, tends to ζ_1 and that this limit ζ_2 is found on the curve Γ_2 :

$$\lim_{z_1 \to \zeta_1,\ z_1 \in G_1} F(z_1) = \zeta_2, \quad \zeta_1 \in \Gamma_1, \quad \zeta_2 \in \Gamma_2.$$

Depending on this, we can extend the definition of the function $F(z_1)$ onto the closed domain $\overline{G_1}$, assuming $F(\zeta_1) = \lim_{z \to \zeta_1,\ z_1 \in G_1} F(z_1)$. The function thus defined will be continuous, in the exended sense, in G, particularly on Γ_1. We assume that this is not so and that $F(z)$ is discontinuous at some point $\zeta_1^{(0)} \in \Gamma_1$. We assume for definiteness that $\zeta_1^{(0)} \neq \infty$ and $F(\zeta_1^{(0)}) \neq \infty$. Then, in any neighbourhood $|z - (\zeta_1^{(0)}| < \varepsilon$ of the point $\zeta_1^{(0)}$, we can find a point $\zeta_1 \in \Gamma_1$ such that $|F - \zeta_1^{(0)}) - F(\zeta_1)| \geqslant \alpha > 0$; here, α is some constant. But, in the same neighbourhood $|z - \zeta_1^{(0)}| < \varepsilon$, we can find a point $z_1 \in G_1$ so close to ζ_1 that $|F(\zeta_1) - F(z_1)| < \frac{\alpha}{2}$ and, hence, $|F(\zeta_1^{(0)}) - F(z_1)| \geqslant \frac{\alpha}{2} > 0$. Since the radius of the ε-neighbourhood is arbitrarily small, we arrive at a contradiction with the definition of $F(\zeta_1^{(0)})$ as $\lim_{z_1 \to \zeta_1^{(0)},\ z_1 \in G_1} F(z_1)$.

In this argument the domains G_1 and G_2 may change their roles. Hence, for the function $z_1 = \Phi(z_2)$, inverse to $z_2 = F(z_1)$, we find that there exists the limit

$$\lim_{z_2 \to \zeta_2,\ z_2 \in G_2} \Phi(z_2) = \zeta_1 \quad (\zeta_2 \in \Gamma_2,\ \zeta_1 \in \Gamma_1).$$

Assuming $\Phi(\zeta_2) = \zeta_1$, we find that $\Phi(z)$ is continuous, in the extended sense, in $\overline{G_2}$ and, in particular, on Γ_2. It can be easily seen that the functions $\zeta_2 = F(\zeta_1)$ and $\zeta_1 = \Phi(\zeta_2)$ are mutually inverse. With the

help of these functions we can establish between the Jordan curves Γ_1 and Γ_2 a correspondence which is one-to-one and, in the extended sense, continuous on both sides.

Thus, the following *theorem of correspondence between boundaries* holds. It states that *the function $z_2 = F(z_1)$, which maps the domain G_1 bounded by closed Jordan curve Γ_2 conformally onto the domain G_1 also bounded by a closed Jordan curve Γ_1, can, evidently, be defined uniquely at the points of Γ_1 in such a way that it is continuous (in the extended sense) in the closed domain G_1 and effects a one-to-one mapping, which is also continuous (in the extended sense) on both sides of Γ_1 onto Γ_2. In short, under conformal mapping of two domains, bounded by closed Jordan curves, onto each other, a one-to-one and continuous mutual correspondence can always be established between their boundaries.*

As already stated, we would refrain from giving a proof of this theorem because of its complexity.[1] The inverse theorem appears to be more elementary. This is true, however, only under certain limitations. We formulate this for a very simple case, when the Jordan curves Γ_1 and Γ_2 do not pass through the point at infinity and the domains G_1 and G_2 are respectively the interiors of these curves.

Converse of the theorem of correspondence between boundaries. *Let the function $z_2 = F(z_1)$, which is continuous in the closed domain $\overline{G_1}$, bounded by the Jordan curve Γ_1, and is analytic in the domain G_1, effect a mapping (which is one-to-one and continuous on both sides) of Γ_1 onto a certain Jordan curve Γ_2. Then, this function will be single-sheeted in G_1 and will map this domain onto the interior G_2 of the curve Γ_2.*

This theorem can easily be extended also to the case when G_1 is the exterior of the curve Γ_1 or when Γ_1 is a generalised Jordan curve and G is one of the two domains bounded by it (for instance, when Γ_1 is a straight line and G_1 a half-plane). To reduce each of these cases to that mentioned in the theorem, it is sufficient to fix any point $z_1^{(0)}$ outside the domain G_1 and carry out the subsidiary transformation $z' = \dfrac{1}{z_1 - z_1^{(0)}}$. As a result of this, Γ_1 is mapped onto a certain Jordan curve Γ' in the z'-plane (not passing through the point $z' = \infty$), the domain G_1 onto the domain G' (the interior of the curve Γ') and the function $F(z_1)$ onto the function $F\left(z_1^{(0)} + \dfrac{1}{z'}\right)$ which is continuous in the closed domain $\overline{G'}$, analytic in G' and also effects a mapping, which is one-to-one and continuous on both sides, of Γ' onto Γ_2. We note, however, that the

[1] See, A.I. Markushevich, loc. cit., Chapter 5, Art. 3.

curves Γ_1 and Γ_2 play different roles under the conditions of the theorem and the theorem holds even when Γ_2 passes through the point at infinity. The simplest example which corroborates the above statement is the function $z_2 = z_1^3$, as studied in the upper half-plane. It is obvious that it maps the boundary of the half-plane (the real axis), one-to-one and also mutually continuously (in the extended sense of the word), onto the real axis. However, it is not single-sheeted in the upper half-plane and maps the latter not onto the upper half-plane but onto a Riemann surface which can be formed out of the two copies G_2' and G_2''' of the upper half-plane and one copy G_2'' of the lower half-plane, by pasting together G_2' and G_2'' along the negative part of the real axis and G_2'' with G_2''' along the positive part of the same axis (Fig. 62).

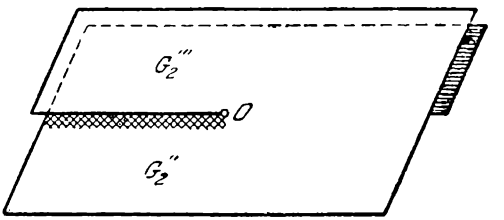


Fig. 62.

Now, we proceed to prove the theorem. For this, we take recourse to the principle of the argument (Art. 2, Chapter 8). Since this principle has been proved under certain well known limitations, we introduce here certain simplifying assumptions, e.g. we assume that Γ_1 is a rectifiable curve and that this curve (along with the domain G_1) belongs to a certain domain G_1' in which $f(z_1)$ is one-valued and analytic. In other words, we now assume that $f(z_1)$ is analytic not only within Γ_1 but also on the curve Γ_1 itself. Let $z_2^{(0)}$ be any point of the domain G_2. We will show that this point belongs to the set of values of $f(z_1)$ in the domain G_1. In this case the equation $f(z_1) - z_2^{(0)} = 0$ has one and only one root in G_1. According to the principle of the argument, the number N of the roots of this equation is equal to $\frac{1}{2\pi} \operatorname*{Var}_{z_1 \in \Gamma_1} \operatorname{Arg} [f(z_1) - z_2^{(0)}]$, on the assumption that z_1 traverses along Γ_1 in the positive direction once. But, by the conditions of the theorem, $z_2 = f(z_1)$ must traverse once in some direction over the entire curve Γ_2. Consequently, the vector $f(z_1) - z_2^{(0)} = z_2 - z_2^{(0)}$, with its origin at the point $z_2^{(0)}$, lying in the interior of Γ_2 and with its end on Γ_2, must rotate through an angle $\pm 2\pi$; from this, we get

$$N = \frac{1}{2\pi} \operatorname*{Var}_{z_1 \in \Gamma_1} \operatorname{Arg} [f(z_1) - z_2^{(0)}] = \frac{1}{2\pi} \operatorname*{Var}_{z_2 \in \Gamma_2} \operatorname{Arg} (z_2 - z_2^{(0)}) = \pm 1.$$

The minus sign is, obviously, excluded ($N \geqslant 0$). Hence, we conclude that the point $z_2 = f(z_1)$ must necessarily move along Γ_2 in the positive direction (with respect to the interior of Γ_2) and the equation $f(z_1) - z_2^{(0)} = 0$

has a single root in the domain G_1. Exactly in the same manner, we can see that for the point z_2', lying outside Γ_2, the equation $f(z_1) - z_2' = 0$ does not have any root at all in the domain $G_1\left(\frac{1}{2\pi}\operatorname*{Var}_{z_1\in\Gamma_1}\operatorname{Arg}[f(z_1)-z_2']=0\right)$.

We have thus proved that the set of values of $f(z_1)$ in the domain G_1 contains all points, lying within Γ_2, and does not contain any point whatsoever from the exterior of Γ_2. Since this set is a domain (each point in it is an interior point with respect to it), it coincides with the domain G_2 bounded by the curve Γ_2. Finally, every point $z_2 \in G_2$ is a value of $f(z_1)$ which it takes up only at one point of the domain G_1. Hence, $z_2 = f(z_1)$ is single-sheeted in the domain G_1 and maps it conformally onto G_2.

We have thus proved a theorem which is less general than the one formulated on page 329. The case when G_1 is the upper half-plane and the function $f(z_1)$ has a finite number of singularities on the real axis, remaining continuous in $\bar{G}_1$, is of great practical importance. Since this case is not covered just by what has been proved above, we shall show how to establish the validity of the theorem for this case. We put $\zeta_1^{(0)} = \infty$ and let $\zeta_1', \ldots, \zeta_1^{(p)}$ be the finite singularaties of the function $f(z_1)$ (lying on the real axis) and $\zeta_2^{(0)}, \zeta_2', \ldots, \zeta_2^{(p)}$ the images of these points (lying on Γ_2). For each natural number n, we draw semi-circles $\lambda_1^{(j)}(n)$ with the points $\zeta_1^{(j)}$ $(1 \leqslant j \leqslant p)$ as their centres and of radii equal to $1/n$. We also draw the semi-circle $\gamma_1^{(0)}(n)$ of radius n and with the origin of coordinates as its centre. When n is sufficiently large $(n > N_0)$, the semi-circle bounded by $\gamma_1^{(0)}(n)$ and the corresponding interval of the real axis will contain all the semi-circles bounded by $\gamma_1^{(j)}(n)$ $(1 \leqslant j \leqslant p)$ and the corresponding intervals $\delta_1^{(j)}(n)$ of the real axis, the latter semi-circles lying (pairwise) one in the exterior of the other. We denote by $D_1(n)$ the domain bounded by all these semi-circles and the

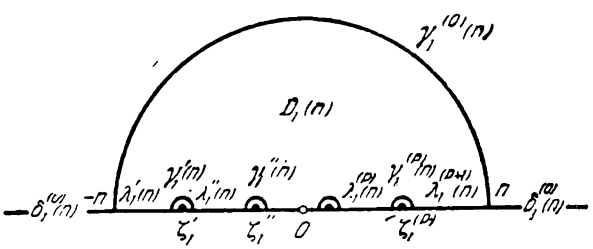


Fig. 63.

intervals $\lambda_1^{(k)}(n)$ $(k=1, 2, \ldots, p+1)$ of the real axis connecting them (Fig. 63); we denote its boundary by $\Gamma_1(n)$. Let, finally, $\delta_1^{(0)}(n)$ be an

infinite interval of the real axis which connects the point n with the point $-n$ through infinity. Evidently, $z_2 = f(z_1)$ is analytic in the domain $D_1(n)$ and on its boundary $\Gamma_1(n)$, and maps $\Gamma_1(n)$ onto a certain continuous curve $\Gamma_2(n)$ (regarding which we do not know *a'priori* whether it would be a Jordan curve or not). $\Gamma_2(n)$, is obtained from Γ_2 by certain deformations of this curve in the neighbourhood of the points $\zeta_2^{(l)}$ $(l=0, 1, \ldots, p)$, such that each arc $\delta_2^{(l)}(n)$, which is an image of the interval $\delta_1^{(l)}(n)$, is replaced by the image $\gamma_2^{(l)}$ of the corresponding semi-circle (Fig. 64).

By virtue of the continuity of the function $f(z_1)$ in the closed domain $\overline{G_1}$, the arcs $\delta_2^{(l)}(n)$ and $\gamma_2^{(l)}(n)$ must contract to the points $\zeta_2^{(l)}$ $(l=0, 1, \ldots, \ldots, p)$ as $n \to \infty$. Hence, for a system of neighbourhoods $u^{(l)}$, as small as desired, of the points $\zeta_2^{(l)}$, we can find a number N such that for $n > N$ the above-mentioned arcs will lie (pairwise) in the neighbourhoods $u^{(l)}$.

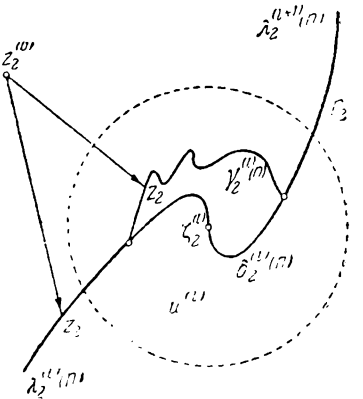


Fig. 64.

Let $z_2^{(0)}$ be an arbitrary point in the domain G_2. We choose the neighbourhoods $u^{(l)}$ of the points $\zeta_2^{(l)}$ so small that $z_2^{(0)}$ lies in the exterior of any of these neighbourhoods, and we choose $N > N_0$ so large that the arcs $\delta_2^{(l)}(n)$ and $\gamma_2^{(l)}(n)$, for $n > N$, are included in the interior of $u^{(l)}$ $(l=0, \ldots, p)$. We shall now show that the equation $f(z_1) - z_2^{(0)} = 0$ has in that case one and only one root within $D_1(n)$. For this, it is sufficient to prove that $\underset{z_1 \in \Gamma_1(n)}{\mathrm{Var}\,\mathrm{Arg}}\,[f(z_1) - z_2^{(0)}]$ (when z_1 traverses $\Gamma_1(n)$ in the positive direction), which evidently coincides with $\underset{z_2 \in \Gamma_2(n)}{\mathrm{Var}\,\mathrm{Arg}}\,(z_2 - z_2^{(0)})$, is equal to 2π.

We shall do this by establishing that

$$\underset{z_2 \in \Gamma_2(n)}{\mathrm{Var}\,\mathrm{Arg}}\,(z_2 - z_2^{(0)}) = \underset{z_2 \in \Gamma_2}{\mathrm{Var}\,\mathrm{Arg}}\,(z_2 - z_2^{(0)}) \quad (n > N).$$

Now, any pair of the arcs $\gamma_2^{(l)}(n)$ and $\delta_2^{(l)}(n)$ by which the curves $\Gamma_2(n)$ and Γ_2 are distinguished from each other have common termini. Hence, $\underset{z_2 \in \gamma_2^{(l)}(n)}{\mathrm{Var}\,\mathrm{Arg}}\,(z_2 - z_2^{(0)})$ may differ from $\underset{z_2 \in \delta_2^{(l)}(n)}{\mathrm{Var}\,\mathrm{Arg}}\,(z_2 - z_2^{(0)})$ only by an integral multiple of 2π. But each of these numbers is individually less than

π in absolute magnitude, because the terminal point of the vector $z_2-z_2^{(0)}$ lies within the circle $u^{(l)}$ while its initial point lies outside $u^{(l)}$ (Fig. 64). Hence

$$\operatorname*{Var}_{z_2\in\gamma_2^{(l)}(n)} \operatorname{Arg}(z_2-z_2^{(0)}) = \operatorname*{Var}_{z_2\in\delta_2^{(l)}(n)} \operatorname{Arg}(z_2-z_2^{(0)} \quad (l=0, \ldots, p)$$

and

$$\operatorname*{Var}_{z_2\in\Gamma_2(n)} \operatorname{Arg}(z_2-z_2^{(0)}) = \operatorname*{Var}_{z_2\in\Gamma_2} \operatorname{Arg}(z_2-z_2^{(0)}) = 2\pi,$$

as stated above.

Thus, each value $z_2^{(0)} \in G_2$ is taken up by the function $f(z_1)$ once in the domain $D_1(n) \subset G_1$, for all sufficiently large values of n. Taking the point z_2' in the exterior of the curve Γ_2, we can, by the same argument, establish that, for $n > N'$,

$$\operatorname*{Var}_{z_2\in\Gamma_2(n)} \operatorname{Arg}(z_2-z_2') = \operatorname*{Var}_{z_2\in\Gamma_2} \operatorname{Arg}(z_2-z_2') = 0,$$

i.e. the value z_2' is not taken up by $f(z_1)$ at any point whatsoever of the domain $D_1(n)$ and, consequently, is also not taken up anywhere in the domain G_1.

Hence it follows that the set of values of the function $f(z_1)$ in the domain G_1 coincides with the domain G_2, $z_2 = f(z_1)$ being single-sheeted in G_1.

We have shown while proving the converse theorem that the correspondence between the curves Γ_1 and Γ_2 preserves the sense of direction of the curves, i.e. the positive direction passes over into the positive. A similar supplementary statement can be made with regard to the fundamental (direct) theorem of the correspondence between boundaries. One can say that *the correspondence between boundaries, which can be established under conformal mapping of the domain G_1 onto the domain G_2, is always such that the point $z_2 = F(z_1)$ describes the boundary of the domain G_2 in the positive direction if z_1 describes the boundary of the domain G_1 in the positive direction*, [i.e. in a direction with respect to which the domain lies to the left of an observer that moves with the point z_1 along the boundary or, more precisely, moves in a direction along which $\operatorname*{Var}_{z_1\in\Gamma_1} \operatorname{Arg}(z_1-z_1^{(0)}) = 2\pi$, for any arbitrary point $z_1^{(0)} \in G_1$].

On the basis of the correspondence between boundaries, we can subject the function $z_2 = f(z_1)$, which maps G_1 conformally onto G_2, to supplementary conditions of a type different from those mentioned in Art. 6. For instance, we may require that during the mapping the point

$z_1^{(0)} \in G_1$ is translated into the point $z_2^{(0)} \in G_2$ and further that the boundary point $\zeta_1^{(0)}$ is translated into the boundary point $\zeta_2^{(0)}$, nothing being possible to predict of Arg $f'(z_1^{(0)})$. Similarly, we may also require that any three points ζ_1', ζ_1'', ζ_1''' of the curve Γ_1 are translated into any three points ζ_2', ζ_2'', ζ_2''' of the curve Γ_2, with the only limitation that the sense of the direction of traversal is preserved. To establish the validity of the last two statements it is sufficient to examine the case when the domain G_2 is, for example, the interior of a unit circle. If a certain conformal mapping of G_1 onto G_2 does not satisfy the conditions imposed (for example, the images of the points ζ_1', ζ_1'', ζ_1''' on the unit circle do not coincide with the pre-assigned ones) the only alternative that remains is to undertake the proper conformal mapping of the unit circle onto itself so as to obtain the required result.

8. MAPPING OF THE UPPER HALF-PLANE BY THE ELLIPTIC INTEGRAL

As an example of the applications of the theorem inverse to that of the correspondence between boundaries, we shall examine the mapping of the upper half-plane by the function

$$w = f(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where $0 < k^2 < 1$ (*elliptic integral of the first kind in the normal Legendre form*). The function $\sqrt{(1-t^2)(1-k^2t^2)}$ is two-valued and has branch points of the first order at $t = \pm 1$ and $t = \pm \frac{1}{k}$. Since all these points lie on the real axis, we can find, in the upper half-plane, two one-valued analytic branches of this function, having opposite values at each point. Of these branches, we choose the one which takes up real positive values on the interval (0, 1) of the real axis. Then, our integral can represent in the upper half-plane a one-valued analytic function which is continuous in the closed half-plane and analytic everywhere except at the points ± 1 and $\pm \frac{1}{k}$ (the integrand can be expanded in the neighbourhood $|z| > \frac{1}{|k|}$ of the point at infinity into the series

$$(1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}} = k^{-1}t^{-2}(1-t^{-2})^{-\frac{1}{2}}(1-k^{-2}t^{-2})^{-\frac{1}{2}} =$$
$$= \pm k^{-1}t^{-2}\left(1+\frac{1}{2}t^{-2}+\ldots\right)\left(1+\frac{1}{2k^2}t^{-2}+\ldots\right) =$$
$$= \pm\left[k^{-1}t^{-2}+\frac{1}{2}(k^{-1}+k^{-3})t^{-4}+\ldots\right];$$

thus, for $f(z)$, we obtain the expression

$$f(z) = C_0 \mp \left[k^{-1}z^{-1} + \frac{1}{2\cdot 3}(k^{-1} + k^{-3})\, z^{-3} + \ldots\right],$$

whence we find that both the branches of the function $f(z)$ are analytic at the point at infinity).

Taking the upper half-plane as the domain G_1 and the real axis as the curve Γ_1 (we have already noted above that an extension of the theorem already proved to the case of unbounded domains and curves is permissible), we study the mapping of the curve Γ_1 by the function $w = f(z)$. As $z = x$ runs over the interval $0 < x < 1$, the function

$$w = f(x) = \int_0^x \frac{dt}{\sqrt[+]{(1-t^2)(1-k^2t^2)}},$$

remaining real, increases from zero to

$$K = \int_0^1 \frac{dt}{\sqrt[+]{(1-t^2)(1-k^2t^2)}}.$$

The integrand here takes the form

$$\frac{1}{\pm i\sqrt[+]{(t^2-1)(1-k^2t^2)}},$$

in the segment $1 < x < \frac{1}{k}$. The sign in the last expression cannot be chosen arbitrarily. It has to agree with the choice made above of the branch of the square root so that the continuity of this branch in the upper half-plane be ensured. Writing out $(1-t^2)(1-k^2t^2)$ in the form

$$k^2(t-1)[t-(-1)]\left(t-\frac{1}{k}\right)\left[t-\left(-\frac{1}{k}\right)\right] = \varphi(t),$$

we find that as we pass from the points of the segment $(0, 1)$ to those of the segment $\left(1, \frac{1}{k}\right)$ along the semi-circle with centre at 1 and belonging to the upper half-plane, the change in Arg $\varphi(t)$, which is made up of the changes in the arguments of the individual factors, is equal to $-\pi$ [actually, Arg $(t-1)$ is reduced by π while the arguments of the

remaining factors do not change]. Thus, Arg $\sqrt{\varphi(t)}$ changes by $-\frac{1}{2}\pi$ and, consequently, takes up the value $-\frac{1}{2}\pi + 2k\pi$. We thus note that out of the two values $\pm i\sqrt{(t^2-1)(1-k^2t^2)}$ we have to choose the value $-i\sqrt{(t^2-1)(1-k^2t^2)}$. Thus, for $1 < x < \frac{1}{k}$, we have

$$w = f(x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} +$$

$$+ i\int_1^x \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = K + i\int_1^x \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}.$$

It follows that as $z = x$ runs over the segment $1 < x < \frac{1}{k}$ the point w runs over a rectilinear segment, parallel to the imaginary axis, from the point K to the point $K + iK'$, where

$$K' = \int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}.$$

This last integral can be represented in a form similar to K by the substitution

$$t = \frac{1}{\sqrt{1-k'^2t'^2}}, \quad \text{where} \quad k'^2 = 1 - k^2.$$

Thus, we obtain

$$K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

We now pass on from the segment $0 < x < \frac{1}{k}$ to the segment $\frac{1}{k} < x < +\infty$. Here, the argument of the expression under the square root,

$$(t^2-1)(1-k^2t^2) = -k^2(t-1)(t+1)\left(t-\frac{1}{k}\right)\left(t+\frac{1}{k}\right),$$

changes, getting reduced by π $\left[\text{along with } \operatorname{Arg}\left(t-\frac{1}{k}\right)\right]$. Hence, for

$\sqrt{(t^2-1)(1-k^2t^2)}$ we obtain a value having the argument $-\frac{\pi}{2}$, viz.

$$-i\sqrt[+]{(t^2-1)(k^2t^2-1)}.$$

Thus, for $\frac{1}{k} < x < +\infty$,

$$w = f(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} +$$

$$+ i\int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} - \int_{\frac{1}{k}}^{x} \frac{dt}{\sqrt{(t^2-1)(k^2t^2-1)}} =$$

$$= K + iK' - \int_{\frac{1}{k}}^{x} \frac{dt}{\sqrt[+]{(t^2-1)(k^2t^2-1)}}.$$

If x increases from $\frac{1}{k}$ to $+\infty$, the integral $\int_{\frac{1}{k}}^{x} \frac{dt}{\sqrt{(t^2-1)(k^2t^2-1)}}$ increases from zero to

$$\int_{\frac{1}{k}}^{\infty} \frac{dt}{\sqrt[+]{(t^2-1)(k^2t^2-1)}} = \int_0^1 \frac{d\tau}{\sqrt[+]{(1-\tau^2)(1-k^2\tau^2)}} = K$$

$\left(\text{we have made the substitution } t = \frac{1}{k\tau}\right)$. Hence, the point $w = f(x)$ describes a rectilinear segment, parallel to the real axis, from the point $K + iK'$ to the point iK'.

Similarly we find that as x describes the segments from zero to -1, from -1 to $-1/k$ and from $-1/k$ to $-\infty$, the point $w = f(x)$ describes successively the rectilinear segments from zero to $-K$, from $-K$ to $-K + iK'$ and, finally from, $-K + iK$ to iK'.

Thus, *the function* $w = f(z)$ *maps the real axis* Γ_1 *one-to-one onto the contour* Γ_2 *of the rectangle with vertices at* $-K, K, K + iK'$ *and*

$-K+iK'$, where

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\quad K'=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$$

and $k'^2=1-k^2$.

Hence, by the foregoing theorem, it follows that this function is single-sheeted in the upper half-plane and maps it conformally onto the foregoing rectangle.

The base of the rectangle is $2K$ and the height K'; hence

$$\frac{K'}{2K}=\frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}}{2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}=\lambda(k).$$

As the parameter k $(0<k<1)$ increases from zero to unity, the denominator of the fraction increases from $2\int_0^1 \frac{dt}{\sqrt{1-t^2}}=\pi$ to ∞, and at the same time $k'\left[k'=\sqrt[+]{1-k^2}\right]$ decreases from unity to zero; hence, the numerator of the fraction decreases from ∞ to $\pi/2$. Thus, the ratio $\frac{K'}{2K}=\lambda(k)$, changing continuously, decreases from ∞ to zero as k increases from zero to unity. So, for any rectangle with base $2a$ and height b, we can find one and only one value of k $(0<k<1)$, for which

$$\frac{b}{2a}=\frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}}{2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}=\frac{K'}{2K}.$$

Consequently, the function

$$f(z)=\int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

constructed for this value of k, maps the upper half-plane conformally onto a rectangle similar to the given one.

If we desire to obtain a mapping onto the given rectangle, it would be sufficient to introduce further a multiplier $\mu = \frac{2a}{2K} = \frac{b}{K'}$. We would then have

$$w = \mu \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

As a result of this mapping, we would obtain a rectangle with vertices at $-\mu K = -a$, $\mu K = a$, $\mu K + i\mu K' = a + bi$ and $-\mu K + i\mu K' = -a + bi$.

Thus, *with the help of the elliptic integral*

$$\mu \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

we can map a half-plane onto a rectangle (with sides of arbitrary lengths) by suitably choosing the values of the parameters k and μ.

9. THE CONCEPT OF THE JACOBIAN ELLIPTIC FUNCTION sn w

Here we discuss certain properties of a function which is inverse to the elliptic integral

$$w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = f(z) \quad (0<k<1).$$

It has been shown in Art. 8 that $w = f(z)$ maps the upper half-plane conformally onto a rectangle Δ_0 with vertices $\pm K$ and $\pm K + iK'$ (Fig. 65). Further, it is found to be continuous in the closed upper half-plane and so maps the real axis, one-to-one and continuously, onto the contour of a rectangle such that the points $0, \pm 1, \pm \frac{1}{k}$ and ∞ are translated into the points $0, \pm K, \pm K + iK'$ and iK', respectively. Hence, it follows that the inverse function, referred to as *the Jacobian elliptic function* and denoted by $z = \text{sn}\,(w\,;\,k)$ or simply by $z = \text{sn}\,w$, is one-valued and single-sheeted in Δ_0, continuous (in the extended sense) in $\overline{\Delta_0}$ and maps the contour of the rectangle, continuously and one-to-one, onto the real axis such that the base AB is transformed into the segment $[-1, +1]$, the lateral sides BC and DA into the segments $[1, 1/k]$ and $[-1/k, -1]$ and, finally, the parts CE and DE of the upper

side into the segments $[1/k, \infty]$ and $[\infty, -1/k]$. It is obvious that the function sn w can be analytically continued through AB, BC, DA, CE and DE according to the Riemann-Schwartz principle of symmetry (Art. 5 of Chapter 9). For example, it may be continued into the rectangle Δ_1: AD_1C_1B, through AB. By the principle of symmetry, the function $z = \text{sn } w$ will take up in this rectangle values that belong to the lower half-plane. Here, by virtue of symmetry in continuation, it will be single-sheeted in Δ_1, the lateral sides BC_1 and AD_1 will be mapped

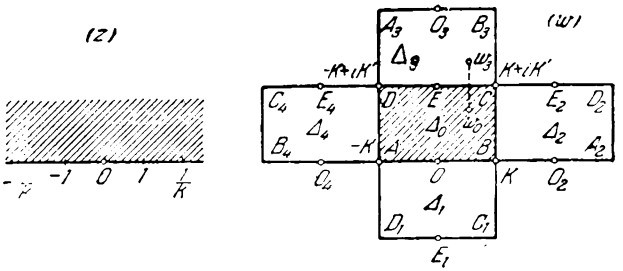


Fig. 65.

onto the segments $[1, 1/k]$ and $[-1/k, 1]$, and C_1E_1 and E_1D_1 onto the segments $[1/k, \infty]$ and $[\infty, -1/k]$. We note that the point 0 is a zero of the function and that it is a simple zero, because the function is single-sheeted in the rectangle DD_1C_1C. A similar statement is also true for analytic continuation through the lateral side BC or DA of the rectangle Δ_0. Since sn w becomes ∞ at the middle point E of the upper side of this rectangle, the continuation may be effected only through the part CE or ED of the upper side. It can be easily seen that in both these cases we obtain a continuation into one and the same rectangle Δ_3, the results of continuation being identical in the two cases. That is, if the point $w_3 \in \Delta_3$ is symmetric with $w_0 \in \Delta_0$ with respect to the straight line on which CE and DE lie, then we obtain for sn w_3, in both the cases, $\text{sn } w_3 = \overline{\text{sn } w_0}$. Thus, the continuation of sn w from Δ_0 through CE or ED leads to the function sn w, which is one-valued and analytic in the rectangle ABB_3A_3 (everywhere, except at the point E, with affix $K'i$, where it becomes ∞). Consequently, $K'i$ is a pole of the function. But sn w is single-sheeted within the rectangle ABB_3A_3; consequently, at the point $K'i$, it has a simple pole (Art. 5). Continuing our function into the rectangles Δ_1, Δ_2, Δ_3, and Δ_4, we can further continue it analytically according to the principle of symmetry into more and more new rectangles. An unrestricted iteration of this process leads to a situation in which the function sn w is found to be continued over the entire finite w-plane, as a single-sheeted and analytic function which does not have

singularities other than the infinite set of poles. Consequently, the elliptic function $z = \operatorname{sn} w$ is a meromorphic function. In this process of continuation the whole plane is covered by a network of equal and identically ordered rectangles which can be mapped conformally by $z = \operatorname{sn} w$ onto the upper or the lower half-plane alternately. We have represented a

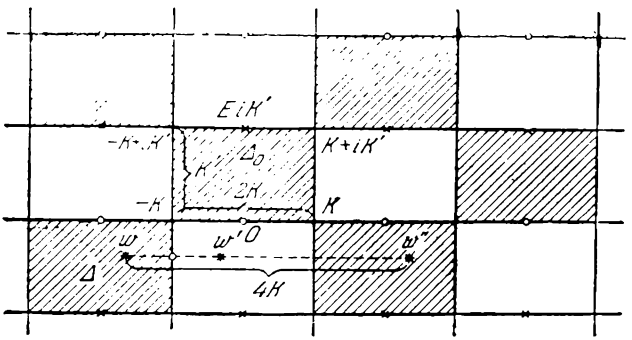


Fig. 66.

part of this network diagrammatically in Fig. 66 and have shaded those rectangles that can be mapped onto the upper half-plane. Further, we have marked the zeros of the function sn w by small circles and its poles by asterisks (the positions of these zeros and poles follow, as has been shown above, immediately from the principle of symmetry and the values of sn w in Δ_0). Evidently, all the zeros of sn w are contained in the expression $2mK + 2niK'$ and all the poles in the expression $2mK + (2n+1)\,iK'$ (m and n are arbitrary integers); all these have the multiplicity 1, i.e. they are simple.

Finally, we show that the function sn w is periodic and has a period of the form $4mK + 2niK'$, where m and n are any integers. It is sufficient to show that the periods here are $4K$ and $2iK'$, because from this it would follow immediately that their integral multiples or the sums thereof will also be the periods. We now turn to Fig. 66. Let w be a point in one of these rectangles. We apply the principle of symmetry twice, each time continuing the function sn w through the right side of the corresponding rectangle. We find that $\operatorname{sn} w' = \overline{\operatorname{sn} w}$ and $\operatorname{sn} w'' = \overline{\operatorname{sn} w'}$, whence $\operatorname{sn} w'' = \operatorname{sn} w$. But it is evident that the segment with its termini at the points w and w'' is parallel to the real axis and is twice in length compared with the base of the rectangle; hence, $w'' = w + 4K$. Thus, $\operatorname{sn}(w+4K) = \operatorname{sn} w$ (for any w). Similarly, applying the principle of symmetry twice such that every time the continuation is effected through the upper side of the corresponding rectangle, we get $\operatorname{sn}(w+2iK') = \operatorname{sn} w$.

We observe that the function $z = \operatorname{sn} w$ for real values of the independent variable $w = u$ is met with in the study of oscillations of a simple pendulum. This problem consists in studying the oscillations in a certain vertical plane (Fig. 67) of a heavy spherical bob hung by a thin thread.

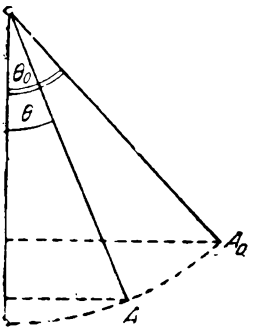


Fig. 67.

Let l be the length of the thread, m the mass of the bob, g the acceleration due to gravity, θ the angle of inclination of the thread from the vertical direction at time t and θ_0 the maximum value of θ. Comparing the position of the bob at the point A_0 where its velocity is equal to zero with that at the point A where its velocity is $v = l\frac{d\theta}{dt}$, we find, according to the energy theorem,

$$\frac{mv^2}{2} = mg(l\cos\theta - l\cos\theta_0).$$

Replacing v here by its value $l\frac{d\theta}{dt}$ and integrating, we obtain

$$t = \sqrt{\frac{l}{2g}}\int_0^{\theta}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\theta}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}};$$

we shall reckon the time t from the instant when $\theta = 0$. We modify the expression under the integral sign by putting $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\tau$ and obtain thereby

$$t = \sqrt{\frac{l}{g}}\int_0^{\tau}\frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2\frac{\theta_0}{2}\tau^2\right)}},$$

whence we get

$$\tau = \operatorname{sn}\left(\sqrt{\frac{g}{l}}t;\ \sin\frac{\theta_0}{2}\right)$$

or

$$\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\cdot\operatorname{sn}\left(\sqrt{\frac{g}{l}}t;\sin\frac{\theta_0}{2}\right).$$

This formula reduces the problem of studying the oscillations of a simple pendulum to that of studying the behaviour of the elliptic function $\operatorname{sn}(u; k)$, for $k = \sin \frac{\theta_0}{2}$. Since the (real) period of an elliptic function is $K = 4 \int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2 \frac{\theta_0}{2} \cdot \tau^2\right)}}$, the period of oscillations of the pendulum will obviously be $4K / \sqrt{\frac{g}{l}}$, i.e.

$$4 \sqrt{\frac{l}{g}} \int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2 \frac{\theta_0}{2} \tau^2\right)}}.$$

We have established for sn w the existence of two *fundamental periods*: $2iK'$ and $4K$. The ratio of these is imaginary. All the remaining periods are linear combinations of these with integral coefficients. Functions, possessing periods with such properties, are called *doubly-periodic* (as distinguished from singly-periodic functions such as sin w and e^w for which any period is an integral multiple of a singlef undamental period). Thus, the Jacobian elliptic function is a meromorphic doubly-periodic function. In general, every meromorphic doubly-periodic function is called *elliptic*. If $2\omega_1$ and $2\omega_2$ are its fundamental periods (the ratio $2\omega_2 : 2\omega_1$ is complex, but not necessarily a purely imaginary number), the parallelogram constructed on the vectors $2\omega_1$ and $2\omega_2$, originating from any point w_0, is called the *period parallelogram* of the elliptic function. For the function $z = \operatorname{sn}(w; k)$, where $0 < k < 1$, the period parallelogram is a rectangle with sides parallel to the coordinate axes and lengths $4K$ and $2K'$ (it contains four such rectangles as we have been speaking of uptil now). The significance of the period parallelogram lies in that by studying the behaviour of the elliptic function in it we can, by the (double-) periodicity principle follow its behaviour in any other parallelogram obtained from the above by a shift through a vector representing any period. Elliptic functions have several applications in the fields of mathematics and mechanics. Hence, their study forms the subject-matter of a special branch of the theory of analytic functions—the theory of elliptic functions. We shall not discuss this theory here and shall rather restrict ourselves to stating one of the fundamental theorems of this theory. The theorem is due to Liouville and states that *if the elliptic function $z = \varphi(w)$ is not identically a constant, then it must have at least one pole in its period parallelogram (inside it or on the sides).*

In fact, if we assume that there are no poles, $\varphi(w)$ will have a bounded modulus in the closed period parallelogram and, consequently, will be analytic and have a bounded modulus over the entire finite plane, i.e. it will be a constant.

10. THE CHRISTOFFEL-SCHWARTZ INTEGRAL

A generalisation of the elliptic integral examined in Art. 8 is provided by the *Christoffel-Schwartz integral*

$$w = f(z) = C \int_0^z (t-a_1)^{\alpha_1-1} (t-a_2)^{\alpha_2-1} \ldots (t-a_n)^{\alpha_n-1}\, dt; \qquad (8)$$

here, C is a positive number (for instance, $C = 1$), $a_1, a_2, \ldots, a_n$ are different real numbers (we assume for definiteness that $a_1 < a_2 < \ldots a_n$) and the exponents $\alpha_1 - 1, \ldots, \alpha_n - 1$ are also real (some of these might be equal). The elliptic integral of Art. 8 can be obtained from this one if we put $n = 4$, $a_1 = -\frac{1}{k}$, $a_2 = -1$, $a_3 = 1$, $a_4 = \frac{1}{k}$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$.

Turning to the general case, we impose on the exponents $\alpha_j - 1$ such restrictions as secure the convergence of the integral at the points a_k and also at the point $z = \infty$. These restrictions must evidently be:

$$\alpha_j - 1 > -1 \ (j=1, 2, \ldots, n) \text{ and } \alpha_1 + \ldots + \alpha_n - n < -1; \qquad (9)$$

[the last one results from the fact that the function under the integral sign can be represented as

$$z^{\alpha_1+\ldots+\alpha_n-n}\left(1-\frac{a_1}{z}\right)^{\alpha_1-1} \ldots \left(1-\frac{a_n}{z}\right)^{\alpha_n-1}$$

in the neighbourhood of the point $z = \infty$].

We now choose a particular one-valued branch of the function $\lambda(z) = (z-a_1)^{\alpha_1-1} \ldots (z-a_n)^{\alpha_n-1}$ in the upper half-plane and subject it to the following specification: on the part $z = x < a_1$ of the real axis, where all the differences $z - a_j$ are negative, we choose for the argument of each of these differences a value equal to π and for the argument of $\lambda(z)$ a value $(\alpha_1-1)\pi + \ldots + (\alpha_n-1)\pi$. Under these conditions, we must give to the argument of $\lambda(z)$; in the intervals

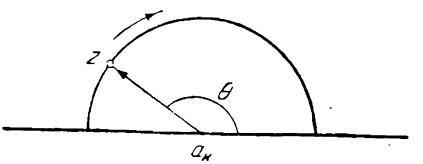


Fig. 68.

(a_{k-1}, a_k), values that agree with those chosen above. Here, it is sufficient to note that when the point z passes from the interval (a_{k-1}, a_k) to the interval (a_k, a_{k+1}) $(k=1, 2, \ldots, n, a_0 = \infty$ and $a_{n+1}=\infty)$ it is only one of the differences $z - a_i$ that changes its sign (from + to −), and this one is the difference $z - a_k$. We now let z describe a semi-circle, with its centre at a_k, belonging to the upper half-plane (Fig. 68) and find that the argument of $z - a_k$ must be reduced by π and, hence, take up the value 0. Accordingly, the argument of the power $(z-a_k)^{\alpha_k-1}$ must change by $-(\alpha_k-1)\pi$ and that of $\lambda(z)$ by the same quantity. Since this argument had in the interval (∞, a_1) the value $(\alpha_1-1)\pi + \ldots + (\alpha_n-1)\pi$, it would, while changing by $-(\alpha_1-1)\pi$ in the interval (a_1, a_2), preserve the constant value $(\alpha_2-1)\pi + \ldots + (\alpha_n-1)\pi$. In general, in the interval (a_{k-1}, a_k) $(k=2, 3, \ldots, n)$ it will have the value $(\alpha_k-1)\pi + \ldots + (\alpha_n-1)\pi, \ldots,$ and, finally, in the interval (a_n, ∞) the value 0. Putting, for the sake of brevity, $C\int_0^{a_{k-1}} (t-a_1)^{\alpha_1-1} \ldots (t-a_n)^{\alpha_n-1}\, dt = w_k$, we examine the behaviour of the integral (8) in the interval (a_{k-1}, a_k). We have

$$w = f(z) = w_{k-1} + C\int_{a_{k-1}}^{z} \lambda(t)\, dt =$$

$$= w_{k-1} + Ce^{(\alpha_k-1)\pi i + \cdots + (\alpha_n-1)\pi i}\int_{a_{k-1}}^{z} |\lambda(t)|\, dt. \qquad (10)$$

The second term on the right-hand side maintains the same argument $(\alpha_k-1)\pi + \ldots + (\alpha_n-1)\pi$, while its modulus increases continuously from zero to $l_k = C\int_{a_{k-1}}^{a_k} |\lambda(t)|\, dt$ as $r = x$ increases from a_{k-1} to a_k. Hence it follows that when z increases in the interval (a_{k-1}, a_k), $w = f(z)$ describes in one and the same sense a rectilinear segment Δ_{k-1} starting from the point w_{k-1}, inclined at an angle $(\alpha_k-1)\pi + , \ldots , + (\alpha_n-1)\pi$ to the positive direction of the real axis, having a length equal to l_k and ending at the point $w_k = C\int_0^{a_k} (t-a_1)^{\alpha_1-1} \ldots (t-a_n)^{\alpha_n-1}\, dt$. This result holds in the interval (∞, a_1) (here, $w_0 = C\int_0^{\infty} (t-a_1)^{\alpha_1-1} \ldots$ $\ldots (t-a_n)^{\alpha_n-1}\, dt$ is the initial point of the segment Δ_0) and also in the

interval (a_n, ∞) (here, $w_{n+1} = C\int_0^{\infty} (t-a_1)^{\alpha_1-1} \dots (t-a_n)^{\alpha_n-1}\, dt$ is the terminal point of the segment Δ_n). From the above argument it follows that the *function* (8) *maps the real axis onto the polygon* Λ, with segments $\Delta_0, \dots, \Delta_n$ and vertices $w_0, w_1, \dots w_n, w_{n+1} = w_0$. However, it is not possible to confirm that this mapping is one-to-one, because the polygon may intersect itself, i.e. the segments Δ_j may have common points besides the vertices. We assume the absence of self-intersections (as, for instance, in the case of an elliptic integral). Then, Λ is a closed Jordan curve bounding a polygon with vertices $w_0, \dots, w_n$; consequently, according to Art. 7, the Christoffel-Schwartz integral (8) maps the upper half-plane conformally onto the given polygon. From Fig. 69 it follows that the interior angle of this polygon at the vertex w_k $(0<k<n+1)$ is equal to $\alpha_k\pi$. Though the magnitudes of all the angles marked in this figure must be regarded as determined only to the extent of an integral multiple of 2π, we

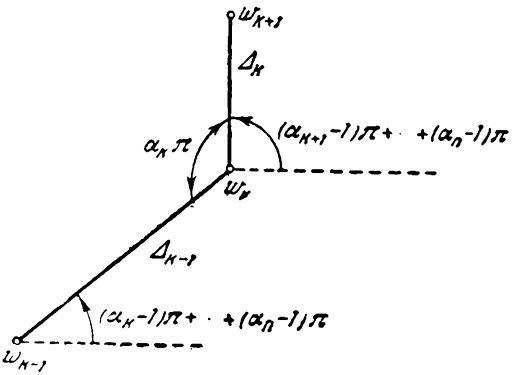


Fig. 69.

can show that $\alpha_k\pi$ gives correctly the interior angle of the polygon. In other words, we have the inequalities $0 < \alpha_k < 2$, only a part of which $(\alpha_k>0)$ was specified by the conditions (9). To prove this, we have to use the property of single-sheetedness of the mapping $w = f(z)$ in the neighbourhood of the point a_k. We have

$$\lambda(z) = [(z-a_1)^{\alpha_1-1} \dots (z-a_{k-1})^{\alpha_{k-1}-1} (z-r_{k+1})^{\alpha_{k+1}-1} \dots \dots (z-a_n)^{\alpha_n-1}] (z-a_k)^{\alpha_k-1},$$

where the expression within the square brackets is a function which is analytic and does not vanish in the neighbourhood of the point a_k.

Consequently, this expression can be expanded into a series in the powers of $(z-a_k)$, with a constant nonzero term : $A_0^{(k)} + A_1^{(k)}(z-a_k) + \ldots$; this series will be uniformly convergent in a certain neighbourhood of the point a_k. But then, in this neighbourhood, we would have

$$w = w_k + C\int_{a_k}^{z}\lambda(t)\,dt = w_k + C\int_{a_k}^{z}[A_0^{(k)}+A_1^{(k)}(t-a_k)+\ldots](t-a_k)^{\alpha_k-1}\,dt =$$

$$= w_k + C\frac{A_0^{(k)}}{\alpha_k}(z-a_k)^{\alpha_k} + C\frac{A_1^{(k)}}{\alpha_k+1}(z-a_k)^{\alpha_k+1} + \ldots \quad (A_0^{(k)}\neq 0),$$

whence it follows that

$$w - w_k = C\frac{A_0^{(k)}}{\alpha_k}(z-a_k)^{\alpha_k}\left[1+\frac{\alpha_k}{\alpha_k+1}\frac{A_1^{(k)}}{A_0^{(k)}}(z-a_k)+\ldots\right].$$

If z tends to a_k along the radius of the circle, with its centre at a_k, that makes an angle θ $(0<\theta<\pi)$ (see Fig. 68) with the positive direction of the real axis, then it follows from the last formula that w will tend to the point w_k along a certain curve, the tangent to which at the point w_k will be inclined at an angle given by $\lim\limits_{w\to w_k}\operatorname{Arg}(w-w_k) =$ $= \operatorname{Arg} A_0^{(k)} + \alpha_k\theta$ (we have taken here $C>0$ and $\alpha_k>0$). If θ changes continuously from zero to π, the radii, originating from the point a_k, will also change continuously, describing a semi-circle with its centre at a_k ; accordingly, their images under conformal mapping—the above-mentioned curves—will describe the whole angle of the polygon and the vertex w_k. But the angle of inclination of the tangents to these curves at the point w_k would change in this case by the quantity $\alpha_k\pi$ (from the value $\operatorname{Arg} A_0^{(k)}$ to the value $\operatorname{Arg} A_0^{(k)} + \alpha_k\pi$) ; hence, $\alpha_k\pi$ is also the magnitude of the interior angle of the polygon and, consequently, $0 < \alpha_k < 2$ $(k=1, 2, \ldots, n)$.

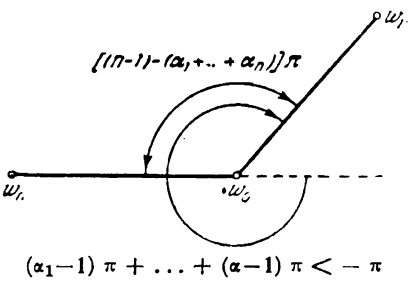


Fig. 70.

For the vertex $w_0 = w_{n+1}$, the magnitude of the interior angle is (see Fig. 70)

$$[(n-1) - (\alpha_1 + \ldots + \alpha_n)]\,\pi.$$

We can conclude from the inequality (9) that this is a positive number. However, if we consider that the given polygon has in all $n+1$

angles and, hence, the sum of all the angles is necessarily equal to $[(n+1)-2]\pi=(n-1)\pi$, we must get for the $(n+1)$-th angle (knowing the magnitudes of the remaining n angles, $\alpha_1\pi, \ldots, \alpha_n\pi$) exactly the foregoing result

$$[(n-1)-(\alpha_1+\ldots+\alpha_n)]\pi.$$

Hence we must have

$$0<[(n-1)-(\alpha_1+\ldots+\alpha_n)]\pi<2\pi,$$

whence

$$\alpha_1+\alpha_2+\ldots+\alpha_n>n-3.$$

Thus, finally, the numbers α_k $(k=1, 2, \ldots, n)$ must satisfy the inequalities:

$$0<\alpha_k<2\ (k=1, \ldots, n),\quad n-3<\alpha_1+\ldots+\alpha_n<n-1. \quad (9')$$

These conditions are necessary in order that the integral (8) may give a conformal mapping of the upper half-plane onto a certain bounded polygon with vertices w_k $(k=0, 1, \ldots, n)$ and interior angles $\alpha_k\pi$ $(k=1, 2, \ldots, n)$ and $[(n-1)-(\alpha_1+\ldots+\alpha_n)]\pi$; however, these conditions are not sufficient.

In the particular case, when $\alpha_1+\ldots+\alpha_n=n-2$, we get the value π for the magnitude of the angle at the vertex w_0. Geometrically, it means that w_0 is not a vertex but simply one of the interior points of the side joining the vertices w_n and w_1. In other words, the integral (8) in this case maps the half-plane not onto a $(n+1)$-gon but onto an n-gon. The same is true of the elliptic integral, in which $n=4$ and $\alpha_1+\alpha_2+\alpha_3+\alpha_4=4\cdot\frac{1}{2}=4-2$.

We shall use the Christoffel-Schwartz integral for mapping the half-plane onto a triangle with angles $\alpha_1\pi$, $\alpha_2\pi$, $\alpha_3\pi$ $(\alpha_k>0, \alpha_1+\alpha_2+\alpha_3=1)$ and a side of length l opposite the angle $\alpha_1\pi$. Here, we can proceed in two different ways. Firstly, we can apply the formula (8) for $n=3$, assigning arbitrarily the real numbers a_1, a_2 and a_3 to the points of the real axis which must be transformed into the three vertices of the triangle (see note at the end of Art. 7). We put, for instance, $a_1=-1$, $a_2=0$ and $a_3=1$. Then, we shall have

$$w=C\int_0^z (t+1)^{\alpha_1-1}\,t^{\alpha_2-1}\,(t-1)^{\alpha_3-1}\,dt.$$

Now, the positive factor C remains to be determined. For this, we note that the length l, by what has been stated above, must be equal to

$$C\int_{0}^{1} |(t+1)^{\alpha_1-1}\, t^{\alpha_2-1}\, (t-1)^{\alpha_3-1}|\, dt = C\int_{0}^{1} (1+t)^{\alpha_1-1}\, t^{\alpha_2-1}\, (1-t)^{\alpha_3-1}\, dt,$$

whence

$$C = l \div \int_{0}^{1} (1+t)^{\alpha_1-1}\, t^{\alpha_2-1}\, (1-t)^{\alpha_3-1}\, dt.$$

Thus

$$w = l\, \frac{\displaystyle\int_{0}^{z} (t+1)^{\alpha_1-1}\, t^{\alpha_2-1}\, (t-1)^{\alpha_3-1}\, dt}{\displaystyle\int_{0}^{1} (1+t)^{\alpha_1-1}\, t^{\alpha_2-1}\, (1-t)^{\alpha_3-1}\, dt}. \tag{11}$$

This function actually solves the problem in hand because it maps the real axis onto a triangle with given angles and sides (the triangle does not have any self-intersection). In the particular case, when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, we obtain an equilateral triangle and the function acquires the form

$$w = l\, \frac{\displaystyle\int_{0}^{z} \frac{dt}{\sqrt[3]{t^2\,(t^2-1)^2}}}{\displaystyle\int_{0}^{1} \frac{dt}{\sqrt[3]{t^2\,(1-t^2)^2}}};$$

for $\alpha_2 = 1/2$, $\alpha_1 = \alpha_3 = 1/3$, we obtain a mapping onto a right-angled isosceles triangle, of side l:

$$w = l\, \frac{\displaystyle\int_{0}^{z} \frac{dt}{\sqrt{t}\,\sqrt[4]{(t^2-1)^3}}}{\displaystyle\int_{0}^{1} \frac{dt}{\sqrt{t}\,\sqrt[4]{(1-t^2)^3}}},$$

and so on.

Secondly, the same problem of mapping the half-plane onto a triangle can be solved by putting $n = 2$ in formula (8) and fixing arbitrarily

two points of the axis a_1 and a_2 which must be translated into the two vertices of the triangle with angles $\alpha_1\pi$ and $\alpha_2\pi$.

Since $(\alpha_1+\alpha_2)$ here is not equal to $n-2\ (=0)$, the point at infinity must be transformed into the third vertex, with an angle equal to $[(2-1)-(\alpha_1+\alpha_2)]\,\pi = \pi - \alpha_1\pi - \alpha_2\pi = \alpha_3\pi$. Choosing, for instance, $a_1 = 0$ and $a_2 = 1$, we obtain, the transformation function in the form

$$w = C\int_0^z t^{\alpha_1-1}(t-1)^{\alpha_2-1}\,dt.$$

For determining C, we obtain the equation

$$C\int_1^\infty |\,t^{\alpha_1-1}(t-1)^{\alpha_2-1}\,|\,dt = l,$$

whence

$$C = \frac{l}{\displaystyle\int_1^\infty t^{\alpha_1-1}(t-1)^{\alpha_2-1}\,dt}.$$

Consequently, the problem in hand will also be solved by the function

$$w = l\,\frac{\displaystyle\int_0^z t^{\alpha_1-1}(t-1)^{\alpha_2-1}\,dt}{\displaystyle\int_1^\infty t^{\alpha_1-\alpha}(t-1)^{\alpha_2-1}\,dt}, \tag{12}$$

which, evidently, is different from (11).

As an example, we obtain, for mapping the half-plane onto an equilateral triangle with sides of length l, the function

$$w = l\,\frac{\displaystyle\int_0^z \frac{dt}{\sqrt[3]{t^2(t-1)^2}}}{\displaystyle\int_1^\infty \frac{dt}{\sqrt[3]{t^2(1-t)^2}}}.$$

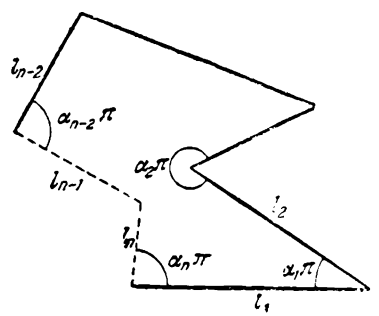


Fig. 71.

In the general case, when the problem is to map it onto an n-gon ($n\geqslant 4$) with angles $\alpha_1\pi, \ldots, \alpha_n\pi$ and sides of lengths $l_1, l_2, \ldots, l_n$ [the angles $\alpha_{k-1}\pi$ and $\alpha_k\pi$ belonging to the side l_k ($k=2, \ldots, n$) and the angles $\alpha_n\pi$ and $\alpha_1\pi$ to the side l_1], we can arbitrarily

choose in formula (8) the points a_1, a_2 and a_3 which must be translated into the vertices w_1, w_2 and w_3 with angles $\alpha_1\pi$, $\alpha_2\pi$ and $\alpha_3\pi$ (*cf.* note at the end of Art. 7). It now remains to determine the further $n-3$ points on the real axis : $a_4, \ldots, a_n$ and the positive factor C; thus, in all, there remain $n-2$ unknowns. At a first glance it appears that for their determination we will have a surplus number of equations, i. e. n equations expressing the lengths of the sides of the polygon in the form of corresponding integrals. It can, however, be easily seen that if the angles of the polygon are known, then knowing only the lengths of the $n-2$ sides $l_1, l_2, \ldots, l_{n-2}$ we can uniquely determine the lengths of the remaining two sides l_{n-1} and l_n (Fig. 71). So there remain only $n-2$ independent equations,

$$C\int_{a_{k-1}}^{a_k} |(t-a_1)^{\alpha_1-1}\ldots(t-a_n)^{\alpha_n-1}|\,dt = l_k \qquad (13)$$

$$(k=1, 2, \ldots, n-2),$$

for determining the $n-2$ unknowns, $a_4, \ldots, a_n$ and C.

On the basis of the Riemann theorem and the theorem on the correspondence between boundaries we can show that the constants in the integral (8) may always be so chosen that they give a mapping of the half-plane onto any pre-assigned polygon. Hence, equation (13) will always have solutions. For details of the mapping of half-planes onto arbitrary polygons we refer the reader to more exhaustive courses on the theory of functions [see, in this connection, A. I. Markushevich, loc. cit., pp. 676—686].

11. FLOW PAST A CIRCULAR CYLINDER (WITHOUT CIRCULATION)

Finally, we give some applications of the theory of analytic functions to hydromechanics.[1]

We investigate the motion of a liquid flowing around a circular cylinder with a velocity $U+iV=Ae^{i\alpha}$ at infinity.

For this purpose, we use the method of conformal mapping. Let $|z|=R$ be a cross section of the cylinder made by the xy-plane (or the projection of the cylinder on this plane). Then, the function $z_1=\dfrac{ze^{-i\alpha}}{R}$ maps conformally the exterior of the circle $|z|>R$ onto the interior of the

[1] See Arts. 13—14 of Chapter 2.

unit circle in the z_1-plane, the vector $Ae^{i\alpha}$ being translated into the vector A/R directed along the real axis (in the positive direction). Further, the function $z_2 = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$ maps the exterior of the unit circle conformally onto the interior of the segment $-1 \leqslant x_2 \leqslant 1$ of the real axis: $y_2 = 0$.

Hence, the function $z = \frac{1}{2}\left(\frac{ze^{-i\alpha}}{R} + \frac{R}{ze^{-i\alpha}}\right)$ is analytic in the exterior G of the projection of the given cylinder, its imaginary part y_2 maintaining a constant value, viz. zero, on the boundary $|z| = R$. It thus follows that if this function is looked upon as a complex flow potential of the liquid in the domain G, the boundary of the domain will be one of the stream lines, i.e. the liquid will flow past the cylinder $|z| = R$.

For the velocity of flow we have the expression

$$\left(\overline{\frac{dz_2}{dz}}\right) = \frac{1}{2}\left(\overline{\frac{e^{-i\alpha}}{R} - \frac{R}{z^2 e^{-i\alpha}}}\right) = \frac{1}{2}\left(\frac{e^{i\alpha}}{R} - \frac{Re^{-i\alpha}}{\bar{z}^2}\right),$$

whence we find that the velocity at the point at infinity is equal to $\frac{1}{2}\frac{e^{i\alpha}}{R}$. This quantity differs from the given one, $Ae^{i\alpha}$, only in the real positive factor $\frac{1}{2AR}$. Multiplying the function constructed above by $2AR$, we finally obtain the function

$$f(z) = Ae^{-i\alpha}z + \frac{Ae^{i\alpha}R^2}{z} = (U - iV)z + \frac{(U+iV)R^2}{z},$$

whose imaginary part has, as before, a constant value (equal to zero) on the circle $|z| = R$ and whose derivative has at the point at infinity the value $U - iV$, conjugate to the given value of the velocity. Thus, the function

$$f(z) = (U - iV)z + \frac{(U+iV)R^2}{z}$$

expresses the complex potential of the flow of the liquid past the cylinder $|z| = R$, with the given velocity $U + iV$ at infinity. For the velocity potential we obtain the expression

$$\varphi(x, y) = \operatorname{Re} f(z) = (Ux + Vy)\left(1 + \frac{R^2}{x^2+y^2}\right),$$

and for the flow function the expression

$$\psi(x, y) = \operatorname{Im}[f(z)] = (-Vx + Uy)\left(1 - \frac{R^2}{x^2+y^2}\right).$$

Then, the equipotential lines have the equations

$$(Ux+Vy)(x^2+y^2+R^2) = C_1(x^2+y^2),$$

and the flow lines

$$(-Vx+Uy)(x^2+y^2-R^2) = C_2(x^2+y^2).$$

These are algebraic curves of the third digree. They are represented diagrammatically in Fig. 72 (for the special case when the velocity at infinity

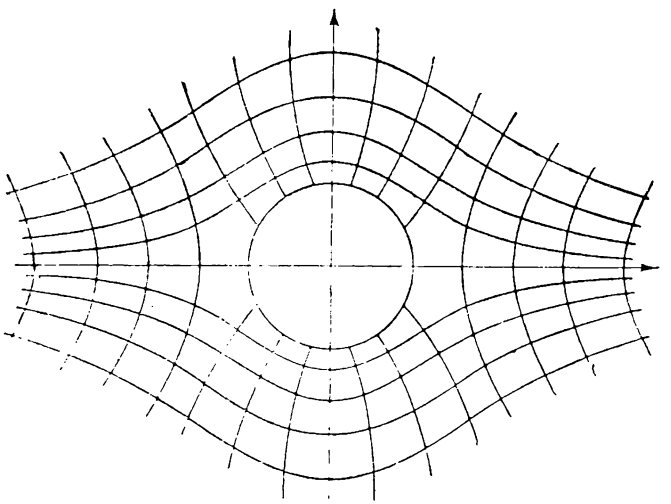


Fig. 72.

is parallel to the real axis). We note that when $C_2 = 0$, we get for the stream lines the straight line $-Vx + Uy = 0$, passing through the origin of coordinates and parallel to the velocity vector at infinity, and the circle $x^2 + y^2 = R^2$. At the intersection points $\pm Re^{i\alpha}$ of these lines the velocity

$$\overline{f'(z)} = Ae^{i\alpha} - \frac{Ae^{-i\alpha}R^2}{\bar{z}^2}$$

vanishes, while at all other points of the plane it is nonvanishing.

We shall see in Art. 14 that it is possible to have flows past the same cylinder with other complex potentials.

12. HYDROMECHANICAL INTERPRETATION OF THE SIMPLEST SINGULARITIES

We now give an interpretation of the simplest singularities of an analytic function as sources, sinks or vortices. We first examine the logarithmic singularity (the branch point of an infinite order).

Let $f(z) = \operatorname{Ln} z$. It is a many-valued function defined in the domain $0 < |z| < \infty$, having a one-valued derivative $f'(z) = 1/z$, and

consequently can be regarded as the complex potential of a certain steady liquid flow. In the given case the velocity potential is one-valued: $\varphi(x, y) = \ln|z|$, while the flow function is many-valued: $\psi(x, y) = \operatorname{Arg} z$. Lines of equal potential $\ln|z| = \text{const}$ or $|z| = \text{const}$ are circles with their centre at the origin of coordinates, whereas the stream lines are rectilinear rays $\operatorname{Arg} z = \text{const}$.

Since $\overline{f'(z)} = \frac{1}{z} = \frac{z}{|z|^2}$ is the velocity at the point z and is, consequently, directed along the ray $\operatorname{Arg} z = \text{const}$ from the origin of coordinates to the point at infinity, all particles move along this direction from the origin of coordinates to the point at infinity with velocities which are larger nearer the origin and smaller farther away from it $\left(|\overline{f'(z)}| = \frac{1}{|z|}\right)$. Fig. 72 suggests us to examine one of the branch points of the function $\operatorname{Ln} z$, viz. the point $z = 0$ as the source of the liquid and the other point $z = \infty$ as its sink. To determine the strength of the source or the sink, we shall calculate the amount of liquid streaming through any arbitrary circle γ with its centre at the origin.

The current of liquid, as determined in Art. 13 (Chapter 2) is

$$\int_{\gamma} -v\,dx + u\,dy = \operatorname{Im}\left[\int_{\gamma} f'(z)\,dz\right] = \operatorname{Im}\left[\int_{\gamma} \frac{dz}{z}\right] = 2\pi.$$

Thus, through a circle of as small a radius as desired, with its centre at the origin of coordinates, there flows per second an amount of liquid equal to 2π (more precisely, we have to assume the height of the surface of a circular cylindrical layer to be unity, the cross section of which is represented by the above circle; the volume of the liquid flowing through this surface is equal to 2π). The number thus obtained will be regarded as the strength of the source $z = 0$, whence the liquid is ejected with an infinitely large velocity or else we can regard this number as the strength of the sink $z = \infty$ where the liquid ceases to flow (or flows with zero velocity).

If instead of the function $\operatorname{Ln} z$, we take the function $F(z) = -i \operatorname{Ln} z$ as the complex potential, the function $\varphi(x, y) = \operatorname{Re}[F(z)] = \operatorname{Arg} z$ will be the velocity potential, the flow function will be $\psi(x, y) = \operatorname{Im}[F(z)] = -\ln|z|$ and the velocity at the point z will be $\frac{i}{z} = \frac{iz}{|z|^2}$. In this case, the equipotential lines will be rectilinear rays emanating from the origin of coordinates, while the flow lines will be circles with their centre at the origin of coordinates.

Since the velocity $\frac{iz}{|z|^2}$ is directed in the positive direction along the tangent to the corresponding circle, each liquid particle moves along the circle (with its centre at the origin of coordinates) circulating around it in the positive (anti-clockwise) direction. As before, the velocity of the particles very close to the origin is very high and that of the particles very far away is very low. Hence, the time required for circulation around the circle is $2\pi r \div \frac{1}{r} = 2\pi r^2$; this time increases proportionately with the square of the radius of the circle. We can easily verify that in this case there are no sources or sinks (this follows from the fact that the flow function in this case is one-valued). As for the velocity circulation along any arbitrary circle γ with its centre at the origin of coordinates, we obtain

$$\int u\,dx + v\,dy = \operatorname{Re}\left[\int f'(z)\,dz\right] = \operatorname{Re}\left[\int_{\gamma} \frac{dz}{z}\right] = 2\pi.$$

Since the quantity of circulation remains the same not only for the circles $|z| = r$ with radii as small as desired but also for circles with radii as large as desired, the origin of coordinates and the point at infinity can be looked upon as the *vortices* of the flow in question and the magnitude 2π can be taken as the intensity of each vortex.

From the examples given above we can easily pass on to the case of a source and a sink of any strength m located at two pre-given points of

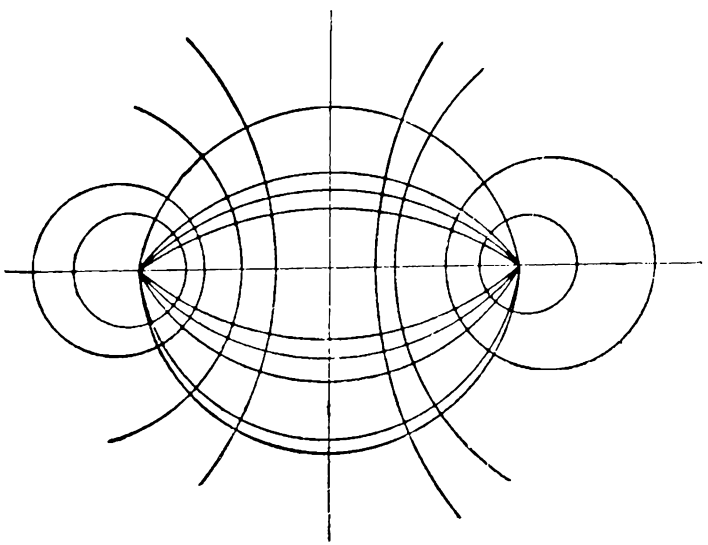


Fig. 73.

the plane or to that of two arbitrarily located vortices with a given intensity Γ. The complex potential $\frac{m}{2\pi} \operatorname{Ln} \frac{z-a}{z-b}$ (a is the source and b the sink)

corresponds to the first case, whereas the complex potential $\frac{\Gamma}{2\pi i}\operatorname{Ln}\frac{z-a}{z-b}$ (a and b being the vortices) corresponds to the second. When the points a and b are given, the general picture of the equipotential lines and of the flow lines is identical in the two cases (Fig. 73). However, in the first case, the lines of flow are arcs of the circles connecting the points a and b, while the equipotential lines are circles orthogonal to them. In the second case, on the other hand, they interchange their role.

We now examine the complex potential which is equal to the sum of the two mentioned above :

$$f(z) = \frac{m-i\Gamma}{2\pi}\operatorname{Ln}\frac{z-a}{z-b}. \tag{14}$$

In this case, the points a and b can be looked upon as vortex-sources i.e. as a combination of a source (or a sink) of strength m and a vortex of intensity Γ. Here, the velocity potential and the flow function are

$$\varphi(x, y) = \frac{m}{2\pi}\ln\left|\frac{z-a}{z-b}\right| + \frac{\Gamma}{2\pi}\operatorname{Arg}\frac{z-a}{z-b}$$

and

$$\psi(x, y) = \frac{m}{2\pi}\operatorname{Arg}\frac{z-a}{z-b} - \frac{\Gamma}{2\pi}\ln\left|\frac{z-a}{z-b}\right|,$$

respectively. The equipotential lines and the flow lines are given by two *mutually orthogonal* families of the so-called *double logarithmic spirals*,

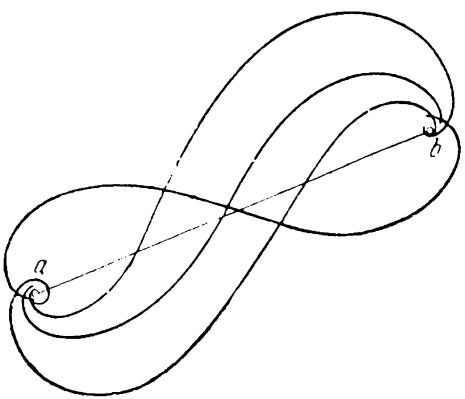


Fig 74.

winding onto the points a and b.[1] Fig. 74 represents diagrammatically three curves of one family and one curve of the other.

[1] These reduce to *logarithmic spirals* through the bilinear transformation $\zeta = \frac{z-a}{z-b}$ and, consequently, form a family of lines intersecting any circular arc connecting the points a and b at one and the same angle.

On the basis of these results we arrive at a hydromechanical interpretation of the pole of an analytic function. Let

$$f(z) = \frac{(m-i\Gamma)}{2\pi} \operatorname{Ln} \frac{z-a}{z-b}$$

be the complex potential, corresponding to two vortex-sources at the points a and b. We write $f(z)$ in the form

$$f(z) = \frac{(m-i\Gamma)(b-a)}{2\pi} \frac{\operatorname{Ln}(z-a) - \operatorname{Ln}(z-b)}{b-a}$$

and assume that b tends to the limit a and $m - i\Gamma$ tends to infinity, such that the product $(m-i\Gamma)(b-a)$ has a limit $Re^{i\alpha}$ which is finite and nonzero. Then, going over to the limit, we obtain the function

$$F(z) = \frac{Re^{i\alpha}}{2\pi} \frac{1}{z-a}, \tag{15}$$

which has a unique singularity, viz. a simple pole at the point a. Thus, a simple pole can be looked upon as a combination of two vortex-sources with the same, indefinitely increasing, intensity.

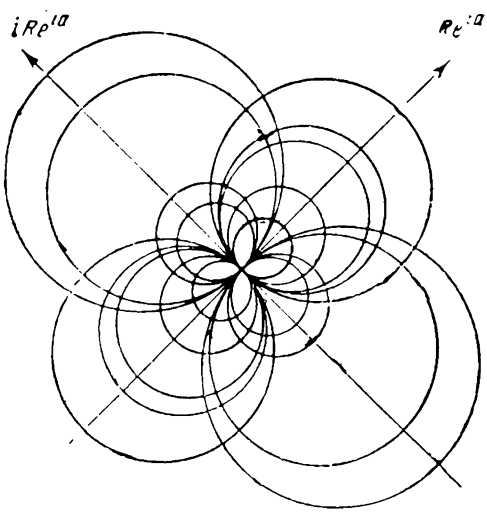


Fig. 75.

The same result can, of course, be obtained if we start either from a source and a sink ($\Gamma=0$) or just from two vortex points ($m=0$).

Putting $z - a = \rho e^{i\theta}$, we write the velocity potential and the flow function in the following form :

$$\Phi(z) = \operatorname{Re} F(z) = \frac{R}{2\pi} \frac{\cos(\alpha-\theta)}{\rho},$$

$$\Psi(z) = \operatorname{Im} F(z) = -\frac{R}{2\pi} \frac{\sin(\alpha-\theta)}{\rho}.$$

It then follows that the equipotential lines and the flow lines,

$$\frac{R}{2\pi} \frac{\cos(\alpha-\theta)}{\rho} = \text{const}$$

and

$$-\frac{R}{2\pi} \frac{\sin(\alpha-\theta)}{\rho} = \text{const},$$

can be represented by two mutually orthogonal families of circles :

$$\rho = C_1 \cos(\alpha-\theta) \text{ and } \rho = C_2 \sin(\alpha-\theta),$$

and in this case the circles of the first family (the equipotential lines) touch, at the point a, the vector $iRe^{i\alpha}$ originating from this point, while the circles of the second family (the flow lines) touch, at the same point, the vector $Re^{i\alpha}$ (Fig. 75).

Similarly, poles of the second order can be interpreted in terms of merging two poles of the first order, poles of the third order in terms of merging two poles of the second order, and so on.

13. GENERAL SOLUTION OF THE PROBLEM OF FLOW PAST A CIRCULAR CYLINDER

We shall now examine the form of the complex potential for the flow of a liquid past a circular cylinder ; this would lead to results of a more general character than the ones obtained in Art. 11.

Suppose we have to find the complex potential of a liquid flow in domain $|z| > R$, assuming that the velocity at the point at infinity is $U + iV$ and that there are no sources, sinks or vortices in the domain $R < |z| < \infty$. Then, as regards the derivative $f'(z)$ of the complex potential, which is conjugate to the velocity at the point z, we find that it must be a one-valued analytic function in the domain $R < |z| < \infty$, taking up the finite value $U - iV$ at the point at infinity. Consequently, the point at infinity is a regular point for it and we obtain

$$f'(z) = U - iV + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \ldots,$$

whence

$$f(z) = (U - iV)\, z + A_1 \operatorname{Ln} z - \frac{A_2}{z} - \frac{A_3}{2z^2} - \ldots .$$

In this case, we have assumed the constant of integration to be zero. In order to obtain from this the flow function $\psi(z) = \operatorname{Im} f(z)$, we have

$$z = re^{i\theta}, \quad A_1 = a_1 + ib_1, \quad A_2 = a_2 + ib_2, \quad A_3 = a_3 + ib_3, \ldots .$$

Hence, we get

$$\psi(re^{i\theta}) = Ur \sin\theta - Vr\cos\theta + a_1\theta + b_1 \ln r + \frac{a_2}{r}\sin\theta - \frac{b_2}{r}\cos\theta +$$

$$+ \frac{a_3}{2r^2}\sin 2\theta - \frac{b_3}{2r^2}\cos 2\theta + \ldots = a_1\theta + b_1 \ln r -$$

$$- \frac{b_2 + Vr^2}{r}\cos\theta + \frac{a_2 + Ur^2}{r}\sin\theta - \frac{b_3}{2r^2}\cos 2\theta + \frac{a_3}{2r^2}\sin 2\theta - \ldots .$$

Since the circle $|z| = R$ is one of the flow lines, the function $\psi(re^{i\theta})$ must preserve its constant value for $r = R$ and for any θ. From the expansion obtained above for $\psi(re^{i\theta})$, which converges for $r > R$, it follows that the stipulated conditions will be satisfied if

$$a_1 = 0, \quad b_2 + VR^2 = 0, \quad a_2 + UR^2 = 0, \quad b_3 = 0, \quad a_3 = 0, \ldots .$$

With the coefficients so chosen, we obtain

$$f(z) = ib_1 \operatorname{Ln} z + (U - iV)\, z + \frac{(U + iV)\, R^2}{z},$$

b_1 being an arbitrary real number.

For the derivative $f'(z)$ we obtain the expression

$$f'(z) = \frac{ib_1}{z} + U - iV - \frac{(U + iV)\, R^2}{z^2},$$

whence

$$\int_{|z| = r} f'(z)\, dz = \int_{|z| = r} \frac{ib_1\, dz}{z} = -2\pi b_1.$$

So the amount of liquid flowing across the circle $|z| = r$ is equal to zero and the velocity circulation along the same circle is equal to $-2\pi b_1$. Choosing b_1 properly, we can obtain any pre-assigned value of the

circulation Γ: $-2\pi b_1 = \Gamma$, whence $ib_1 = \frac{\Gamma}{2\pi i}$. Consequently, $f(z)$ can be written in the form

$$f(z) = \frac{\Gamma}{2\pi i} \operatorname{Ln} z + (U - iV) z + \frac{(U+iV) R^2}{z} + C, \tag{16}$$

where we have introduced an arbitrary constant term C.

Here, the complex potential is represented in the form of a sum of a purely cylindrical flow $\frac{\Gamma}{2\pi i} \operatorname{Ln} z$, corresponding to vortices with intensity Γ at the origin of coordinates and at the point at infinity, and a flow without circulation

$$(U - iV) z + \frac{(U+iV)R^2}{z},$$

derived in Art. 11.

We shall now show that the formula (16) gives a more general solution of the problem of flow past a cylinder, with a given velocity $U + iV$ at the point at infinity and with the given circulation Γ. Let $f_1(z)$ be the complex potential satisfying the same conditions. In this case, the difference $f_1'(z) - f'(z)$, which is conjugate to the difference of velocities of the fluid particles involved in the first and the second motions, is a one-valued analytic function in the domain $|z| > R$. Hence

$$f_1'(z) - f'(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots,$$

whence

$$\int_\gamma [f_1'(z) - f'(z)]\, dz = \int_\gamma \frac{c_1\, dz}{z} = 2\pi i c_1,$$

where γ is an arbitrary closed contour containing in its interior the circle $|z| = R$.

Since the circulations of the two velocities along γ must be equal, we have

$$\operatorname{Re} \int_\gamma [f_1'(z) - f'(z)]\, dz = -2\pi \operatorname{Im} c_1 = 0,$$

i. e. c_1 is a real number. For the difference of the complex potentials, $f_1(z) - f(z)$, we have the expression

$$f_1(z) - f(z) = c_0 + c_1 \operatorname{Ln} z - \frac{c_2}{z} - \frac{c_3}{2z^2} - \ldots$$

and for the difference of their imaginary parts i.e. for the flow functions, we have

$$\delta(r, \theta) = \operatorname{Im}[f_1(z) - f(z)] =$$

$$= \gamma_0 + c_1\theta - \frac{\gamma_2}{r}\cos\theta + \frac{\beta_2}{r}\sin\theta - \frac{\gamma_3}{2r^2}\cos 2\theta + \frac{\beta_3}{2r^2}\sin 2\theta - \ldots,$$

where β_j denote the real and γ_j the imaginary parts of the coefficients c_j. As implied in the problem, the circle $r = R$ must be a flow line for each of the flows examined above ; hence, $\delta(R, \theta) = \text{const} = c$. But, from the expansion derived for $\delta(r, \theta)$, it follows that $\delta(r, \theta) - c_1\theta$ is a one-valued function of $z = re^{i\theta}$. In particular, the function $\delta(R, \theta) - c_1\theta = c - c_1\theta$ will also be a one-valued function of θ ; from this, it follows that $c_1 = 0$.

Thus

$$\delta(r, \theta) = \gamma_0 - \frac{\gamma_2}{r}\cos\theta + \frac{\beta_2}{r}\sin\theta - \frac{\gamma_3}{2r^2}\cos 2\theta + \frac{\beta_3}{2r^2}\sin 2\theta - \ldots.$$

We shall find that $\delta(r, \theta)$ is a one-valued harmonic function in the domain $r > R$ and that it preserves a constant value on the circle $r = R$. It thus follows that $\delta(r, \theta) = \text{const}$ (*cf.* Art. 2) and, consequently, the analytic function $f_1(z) - f(z)$, having $\delta(r, \theta)$ as its imaginary part, is also constant. Thus

$$f_1(z) = f(z) + C' = \frac{\Gamma}{2\pi i}\operatorname{Ln} z + (U - iV)z + \frac{(U + iV)R^2}{z} + C'' ;$$

this completes the proof of the uniqueness of the solution obtained here.

For the velocity potential and the flow function, as determined by the function (16), we have the expressions

$$\varphi(x, y) = \frac{\Gamma}{2\pi}\operatorname{Arg} z + (Ux + Vy)\left(1 + \frac{R^2}{x^2 + y^2}\right) + \beta$$

and

$$\psi(x, y) = -\frac{\Gamma}{2\pi}\ln|z| + (-Vx + Uy)\left(1 - \frac{R^2}{x^2 + y^2}\right) + \gamma,$$

respectively.

Consequently, the equations for the equipotential lines and the flow lines are, respectively,

$$\frac{\Gamma}{2\pi}\operatorname{Arg} z + (Ux + Vy)\left(1 + \frac{R^2}{x^2 + y^2}\right) = C_1$$

and

$$-\frac{\Gamma}{2\pi}\ln|z|+(-Vx+Uy)\left(1-\frac{R^2}{x^2+y^2}\right)=C_2.$$

We have examined these cases for $\Gamma=0$ in Art. 11. When $\Gamma\neq 0$ these are transcendental curves whose form depends on the relative values of Γ and $U+iV$. We assume, for the sake of simplicity, that $V=0$ (we can always arrive at this situation by suitably transforming the coordinate axes) and we find the critical point of flow, i.e. the point at which the velocity vanishes. It follows from the expression for the velocity,

$$\overline{f'(z)}=-\frac{\Gamma}{2\pi i}\frac{1}{\bar{z}}+U-\frac{UR^2}{\bar{z}^2},$$

that these points satisfy the quadratic equation

$$\bar{z}^2-\frac{\Gamma}{2\pi iU}\bar{z}-R^2=0$$

or

$$z^2+\frac{\Gamma}{2\pi iU}z-R^2=0,$$

whence

$$z=\frac{i\Gamma}{4\pi U}+\sqrt{R^2-\left(\frac{\Gamma}{2\pi U}\right)^2}.$$

If $|\Gamma|>4\pi R|U|$, then both the critical points z_1 and z_2 are purely imaginary, and from the relation $z_1z_2=-R^2$ it is evident that only one of these lies outside the circle $|z|=R$, i.e. in the domain occupied by the liquid. The flow lines for this case are represented in Fig. 76. If $|\Gamma|=4\pi R|U|$, we obtain only one critical point, lying on the intersection of the circle $|z|=R$ with the imaginary axis (Fig. 77). Finally, when $|\Gamma|<4\pi R|U|$, there exists, as in the case of a flow without circulation, two critical points lying on the circle $|z|=R$, symmetrically with respect to the imaginary axis (see Fig. 78):

$$\pm\sqrt{R^2-\left(\frac{\Gamma}{4\pi U}\right)^2}+i\frac{\Gamma}{4\pi U}.$$

14. DETERMINATION OF THE LIFT OF AN AEROFOIL

In the previous section we solved the problem of flow past a circle i.e. a circular cylinder. From that study one can, with the help of a conformal mapping, solve the problem of flow past a body of an arbitrary

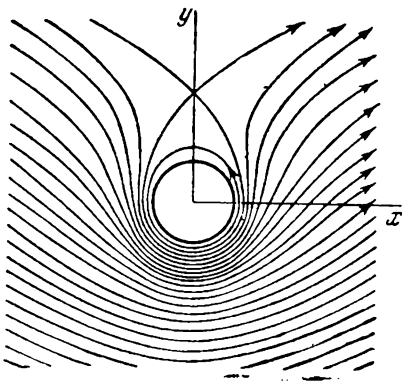


Fig. 76.

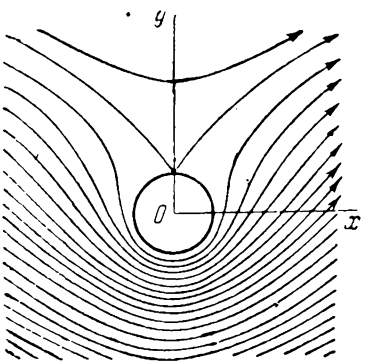


Fig. 77.

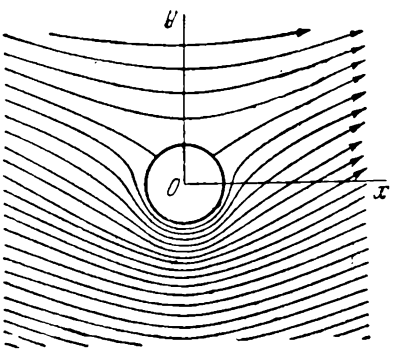


Fig. 78.

form. Let L be a closed Jordan curve in the z-plane. We wish to construct the complex potential of the flow past L with the given velocity $U + iV$ at infinity. We map the exterior of L conformally onto the exterior of the unit circle : $| t | > 1$, such that the point $z = \infty$ is translated into the point $t = \infty$. Let $t = F(z)$ be the function which effects this mapping. In the neighbourhood of the point $z = \infty$ it will have an expansion of the form

$$F(z) = c_1 z + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots,$$

where $c_1 \neq 0$.

We also assume, for the sake of definiteness, that the coefficient c_1 is a real positive number, i.e. $F'(\infty) > 0$. $F(z)$ is defined uniquely by the above-mentioned conditions : (to reduce this case to a mapping of the interior of a Jordan curve onto the interior of a circle, it is sufficient to take the help of the auxiliary mapping $z' = \frac{1}{z-z_0}$, where z_0 is a point lying in the interior of L and $t' = \frac{1}{t}$). Under this mapping, the unknown complex potential $f(z)$ will become the complex potential of flow past a unit circle and, consequently, will have the form

$$f(z) = f[F^{-1}(t)] = \varphi(t) = \frac{\Gamma}{2\pi i} \operatorname{Ln} t + (u-iv)\, t + \frac{u+iv}{t} + c$$

[vide formula (16)]. Since $\varphi'(\infty) = u - iv = \frac{f'(\infty)}{F'(\infty)} = \frac{U-iV}{F'(\infty)}$, the formula obtained above can be rewritten in the form

$$f(z) = \frac{\Gamma}{2\pi i} \operatorname{Ln} F(z) + \frac{U-iV}{F'(\infty)} F(z) + \frac{U+iV}{F'(\infty)\, F(z)} + c. \qquad (17)$$

In this formula, besides the arbitrary constant c which plays no significant role here, there appears the real coefficient Γ. We shall show that we have to choose it equal to the velocity circulation of the flow along any closed curve enclosing the curve L ; for instance, along the image γ_r of the circle $|t| = r > 1$ under the mapping $z = F^{-1}(t)$. In fact,

$$\int_{\gamma_r} f'(z)\, dz = \operatorname*{Var}_{\gamma_r} f(z) = \frac{\Gamma}{2\pi} \operatorname*{Var}_{\gamma_r} \operatorname{Arg} F(z).$$

But when z circulates once along γ_r in the positive direction, $t = F(z)$ also circulates along the circle $|t| = r$ once in the same direction, with $\operatorname*{Var}_{\gamma_r} \operatorname{Arg} F(z) = 2\pi$ and

$$\int_{\gamma_r} f'(z)\, dz = \Gamma,$$

whence follows our statement.

Thus, the flow of the liquid past the contour L is determined by the formula (17), where Γ is the circulation of the flow, $U + iV$ the velocity at the point at infinity and $F(z)$ the function which maps the exterior of the contour L conformally onto the exterior of the unit circle, such that $F(\infty) = \infty$ and $F'(\infty) > 0$.

We shall apply this formula for deriving the complex potential of flow past a *Joukowski-Chaplygin profile* (the profile of an aerofoil with a rounded frontage).

To construct such a profile we examine two circles γ and γ' in the ζ-plane, one of which, say γ, passes through the points ± 1 and touches the other circle γ' from inside at the point 1. Under the mapping $z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$, the circle γ is transformed, as we shall find, into an arc δ of a circle (with its ends at ± 1) and the exterior of γ is conformally mapped onto the exterior of δ (Art. 10 of Chapter 3). Consequently, the circle γ' is mapped one-to-one onto some closed curve δ' belonging to the exterior of δ (save the point $z = 1$ which is common with δ). Since $z = 1$ is an image of the point $\zeta = 1$ and $\frac{dz}{d\zeta} = \frac{1}{2}\left(1 - \frac{1}{\zeta^2}\right) = \frac{\zeta^2 - 1}{2\zeta^2}$ has a simple zero at this point, the angles with vertex at the point $\zeta = 1$ must get doubled by the mapping in question. But the angle between γ and γ', by assumption, is equal to zero. Hence, δ and δ' must also make at the point $z = 1$ an angle equal to zero. The form of the curve δ' is given in Fig. 79.

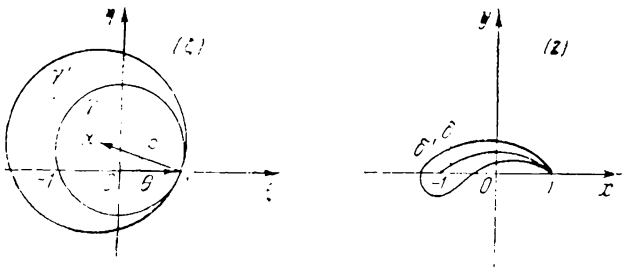


Fig. 79.

This is the Joukowski-Chaplygin profile. Its form and dimensions can be altered first by changing the circles γ and γ' and then by applying a similarity transformation.

To apply the formula (17) for determining the flow past the given profile we have to find a function which maps the exterior of the curve δ' conformally onto the exterior of the unit circle. But the function $z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$ maps the exterior of the circle γ' conformally onto the exterior of the curve δ'. In this case it transforms the point $\zeta = \infty$ into the point $z = \infty$, and its derivative at the point $\zeta = \infty$ has the value 1/2. If the centre of the circle γ' is located at the point α and its radius is equal to ρ, the function $t = \frac{1}{\rho}(\zeta - \alpha)$ would map the exterior of γ' onto the exterior of the unit circle. Thus, the function $z = \frac{1}{2}\left[(\rho t + \alpha) + (\rho t + \alpha)^{-1}\right]$ maps the exterior of the unit circle onto the exterior of δ', such that $t = \infty$ gets transformed into $z = \infty$; its derivative at the point at infinity has the positive value $\frac{\rho}{2}$. Hence, $t = F(z)$ is a function inverse to the function

constructed, i.e. $F(z) = \frac{1}{\rho}\left(-\alpha+z+\sqrt{z^2-1}\right)$; here, we must take that branch of the latter function which approaches infinity at the point at infinity. For this, we have : $F'(\infty) = \frac{2}{\rho}$.

Thus, the unknown complex potential has the form

$$f(z) = \frac{\Gamma}{2\pi i}\operatorname{Ln}\left[\frac{1}{\rho}\left(-\alpha+z+\sqrt{z^2-1}\right)\right] + \\ + \frac{U-iV}{2}\left(-\alpha+z+\sqrt{z^2-1}\right) + \frac{\rho^2(U+iV)}{2\left(-\alpha+z+\sqrt{z^2-1}\right)} + C. \qquad (18)$$

From this it follows that

$$f'(z) = \left[\frac{\Gamma}{2\pi i}\frac{1}{-\alpha+z+\sqrt{z^2-1}} + \right. \\ \left. + \frac{U-iV}{2} - \frac{\rho^2(U+iV)}{2\left(-\alpha+z+\sqrt{z^2-1}\right)^2}\right]\left(1+\frac{z}{\sqrt{z^2-1}}\right).$$

In order that the derivative $f'(z)$, and, consequently, also the velocity, be bounded near the rear edge of the foil, i.e. near the point $z = 1$, it is necessary to introduce the following condition which connects the magnitude of the circulation Γ, the velocity $U + iV$ and the parameters α and ρ, which determine the form of the foil :

$$\frac{\Gamma}{2\pi i}\frac{1}{1-\alpha} + \frac{U-iV}{2} - \frac{\rho^2(U+iV)}{2(1-\alpha)^2} = 0,$$

whence

$$\Gamma = \pi i\left[\frac{\rho^2(U+iV)}{1-\alpha} - (U-iV)(1-\alpha)\right].$$

It is evident from Fig. 79 that $1 - \alpha = \rho e^{-i\theta}$; assuming further that $U + iV = Ae^{i\varphi}$, we obtain

$$\Gamma = -2\pi A\rho\sin(\theta+\varphi). \qquad (19)$$

Consequently, the formula (18) takes the form

$$f(z) = \frac{A\rho}{2}\left\{-\frac{2\sin(\theta+\varphi)}{i}\operatorname{Ln}\left[(\rho e^{-i\theta}-1)+z+\sqrt{z^2-1}\right] + \right. \\ + \frac{e^{-i\varphi}}{\rho}\left[(\rho e^{-i\theta}-1)+z+\sqrt{z^2-1}\right] + \\ \left. + \rho e^{i\varphi}\frac{1}{(\rho e^{-i\theta}-1)+z+\sqrt{z^2-1}}\right\} + C'. \qquad (20)$$

Finally, we calculate the resulting pressure of the flow on the foil (i.e. the lift of the aerofoil). This pressure has to do with a layer of the liquid of unit height ; so, all arguments refer to this layer.

We denote the projections of this force on the coordinate axes by X and Y (since the motion is a plane motion, this force is parallel to the xy-plane). For this, we have the following general formula due to S. A. Chaplygin :

$$X - iY = \frac{id}{2} \int_C [f'(z)]^2 \, dz, \tag{21}$$

where d is the density of the liquid (or gas) and C any closed rectifiable curve containing within it the contour past which the liquid flows.

For calculating the integral (21) it is sufficient to take the residue of the function $[f'(z)]^2$ at the point at infinity. But, in the neighbourhood of the point at infinity, we have

$$[f'(z)]^2 = \frac{A^2\rho^2}{4} \left\{ -\frac{2\sin(\theta+\varphi)}{i} \frac{1}{\rho e^{-i\theta} - 1 + z + \sqrt{z^2-1}} + \frac{e^{-i\varphi}}{\rho} - \right.$$

$$\left. - \rho e^{i\varphi} \frac{1}{(\rho e^{i\theta} - 1 + z + \sqrt{z^2-1})^2} \right\}^2 \left(1 + \frac{z}{\sqrt{z^2-1}}\right)^2 =$$

$$= \frac{A^2\rho^2}{4} \left[\frac{e^{-i\varphi}}{\rho} + \frac{i\sin(\theta+\varphi)}{z} + \ldots \right]^2 \left(2 + \frac{1}{2z^2} + \ldots\right) =$$

$$= A^2 e^{-2i\varphi} + 2iA^2\rho e^{-i\varphi} \sin(\theta+\varphi) \frac{1}{z} + \ldots;$$

Thus, the residue in question is $-2iA^2\rho e^{-i\varphi} \sin(\theta+\varphi)$ and for the lift of the foil we obtain the expression

$$X - iY = -2\pi i A^2 d\rho e^{-i\varphi} \sin(\theta+\varphi),^1$$

or, by the formula (19),

$$X - iY = iAe^{-i\varphi}\Gamma d = i(U - iV)\,\Gamma d$$

and, finally,

$$X + iY = -i(U + iV)\,\Gamma d. \tag{22}$$

Thus, we obtain the famous Joukowski theorem.

[1] We take here the contour C in formula (21) as being traversed in the positive direction i.e. anti-clockwise.

The theorem of Joukowski. *The lift of a foil is orthogonal to the velocity of flow at the point at infinity and is equal in magnitude to the product of this velocity, the (velocity) circulation and the density of the fluid (or gas).*

For further details, we refer the reader to courses on hydromechanics (see, e.g., N. E. Kochin, I. A. Kibel and N. V. Rose, *Theoretical Hydromechanics*, Part 1, or V. V. Golubev, *The Theory of an aerofoil in a Plane Parallel Flow*).

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INDEX

Abel, first theorem of 134
Affix of a point 5
Analytic continuation, chain of 286
— function 2, 30
Angle with vertex at the point at infinity 38
A-point, order of an 190
A-points 190
Argument, the principle of the 258, 261

Bernoulli numbers 202
Bilinear transformation, isometric circle of 37
— —, symmetry-preserving property of 73
Binomial formula 180
Bolzano-Weierstrass theorem 14
Branch point 107
— —, algebraic 107, 312
— —, logarithmic 119, 312
— —, of an infinite order 119, 312
— —, transcendental, of a finite order 312
Broken line 22

Cauchy criterion 9
— integral 174
— — formula 174
— — theorem 146, 179
— — —, generalization of 165
— type integral 267
Cauchy-Hadamard theorem 132
Cauchy-Riemann conditions 28
Chaplygin's formula 367
Christoffel-Schwartz integral 344
Complex plane 6
— —, extended 12
— number 5, 6
— —, argument of 6
— —, degree of 13
— —, geometrical representation of 6
— —, improper 10
— —, modulus of 6
— —, proper 10
— —, the principal value of the argument 6
— —, trigonometric form of 6
— numbers, conjugate 7
— potential 46
Continuation, analytic, *see* analytic continuation
Convergence of series, general criterion for 130
Cross or an harmonic ratio of four numbers or points 69
Cross ratio, invariance of 64, 70
Curve, closed 21
—, continuous 20, 21
—, equation of the 21
—, exterior of 22
—, generalized continuous 23, 24
—, interior of 27
—, multiple points of a 21
—, rectifiable Jordan 181
—, simple (or Jordan) 21
—, smooth 145
Curves, equipotential 48

D'Alembert-Euler equations 28
Derivative 24
—, geometrical interpretation of the argument of 33
—, — — — — modulus of a 35
—, logarithmic 233
Differentiability of a function, necessary and sufficient conditions of 27
Differentiation of functions 26, 27
Dirichlet problem 318
Division of power series, *see* power series
Domain 15
—, convex 285
—, generalized 289
—, multiply-connected 23
— of single-sheetedness of a function 104
—, simply-connected 23, 146
—, — (in the extended plane) 23

Elements of an analytic function 285
Elliptic function 343
— —, Jacobian 339
— integral of the first kind (in the normal form of Legendre) 334

Entire function 52, 234
— —, linear 36
— —, order of an 248
— —, transcendental 87, 237
— —, type of an 248, 249
Equation of continuity of an incompressible fluid 45
Essential singular point 224
Euler constant 247
— numbers 206
Euler's formula 92
Expansion of a meromorphic function into partial fractions 241
— of a rational function into partial fractions 239
—, uniqueness of, into power series 137
Expansions of functions into power series 204, 208
Exponential function 119
— —, general 119
Extended complex plane 12

Flow function 47
— —, characteristic 46
— line 48
— of the fluid across a curve 45
Fresnel integrals 153
Function, analytic 2, 30
—,— at a point at infinity 216
—, — continuation of 283
—, continuous at a point 19
—, — in a set 19
—, —, inversion of 320
—, differentiable relative to a set at a point 24
—, doubly periodic 309
—, elliptic 309
—, fundamental periods of 343
—, —, Jacobian 339
—, entire 52, 236
—, —, linear 36
—, —, transcendental 87, 237
—, exponential 84, 119
—,—, general 119
—exponential, of the complex variable 85
—, fundamental period of 87, 92
—, Gamma 247
—, harmonic 39
—,—, characteristic property of 213
—,—, conjugate 39
—, holomorphic 30
—, hyperbolic 94
—, many-valued, of a complex variable 17
—, meromorphic, expansion of, into partial fractions 241
—, monogenic with respect to a set at a point 24
—, multi-sheeted in a domain 104
—, rational, expansion of, into partial fractions 239
—, single-sheeted in a domain 104
Functions, inverse trigonometric 125
— of a complex variable 17
— — — — —, integral of 141
— — — — — finite order 248
— — the maximal type 249
— — — minimal type 249
— — — normal type 249
—, one-valued branches of many-valued 102, 103
—,—, of a complex variable 17
—, regular 30
—, with general continuity 20
Fundamental theorem of higher algebra 178

Gamma function 247

Harmonic functions 39
— —, characteristic property of 213
— —, conjugate 39
Heine-Borel theorem 14
Holomorphic functions 30
Hyperbolic functions 92

Infinity, the point at 10, 37
Interior point of a set *see* set
Inversion 73
—, local 321
— of an analytic function 320
Isolated singularity of a simple character 222
— — of a multiple character 309

Jacobian elliptic function 339
Jordan curve, rectifiable 267
— theorem 22
Joukowski function 79
—, the theorem of 367, 368

Kernel of the Cauchy integral 211

Laplace equation 40
Laurent's theorem 219
Lift of an aerofoil 363, 367
Limit of a function at a point 19
Limit point of a set 13, 19
Liouville theorem 175, 178, 343
Local criterion of single-sheetedness 318
Logarithm 114
—, the principal value of 115,
—, with an arbitrary base 119
Logarithmic residue of a function 260
Logarithmic spiral 90, 356
— —, double 356
— —, equation of the 90

Many-valued function of a complex variable 17
Mapping, bilinear 58
—,—, circle-preserving property of 61
—,—, symmetry-preserving property of 73
—, conformal at a point 35
—,—, of a domain 35

Mapping conformal, of first kind 35
—,—, of second kind 35
—, identity 59
—, inverse 59
Mappings 60
—, one-to-one and continuous, theorem on 23
Maximum modulus, the principle of 316
Meromorphic function 237, 248
— —, expansion of, into partial fractions 241
— —, Mittag-Leffler expansion of a 241
Method of undetermined coefficients 198, 201
Morera's theorem 182

One-valued branches of a many-valued function 102, 108
— function of a complex variable 17

Period parallelogram of the elliptic function 343
Periodic function, doubly 343
— —, simply 301
Picard, the theorem of 229
Point at infinity 10, 37
Points symmetric with respect to a circle 72
Poisson integral 211
— —, kernel of 211
Pole 206
—, multiple 225
—, simple 225
Polynomial 52
Power function 113
— series 134
—,—, circle of convergence of 133
— —, division of 184
— —, radius of convegence of 133
— —, uniqueness of expansion into 137
Primitive 162
Pringshiem's theorem 301
Profile, Joukowski-Chaplygin 365

Reflection, mirror 73
Regular function 30
— point of an element 296
Residue at the point at infinity 265
— of a function 252
Residues, theorem of 252
ρ-neighbourhood of a point 10
Riemann-Schwartz principle of symmetry 291, 293
Riemann surface of an analytic function 291
Riemann's theorem 326
Rouche's theorem 261

Schwartz formula 213
— lemma 316-317
Sequence, limit of 9
— of numbers converging to infinity 10
— — real numbers, upper limit of 132
Series 130
—, absolute convergence of 132
—, complex term, conversion of 130
—, convergence of, general criterion for 130
—, Laurent's 215
—,—, principal part of 230
—,—, regular part of 231
—, power 134
—, sum of 130
—, Taylor 137
Set, boundary of a 15
—, closed 14
—,—, bounded 14, 20
— distance of a point from 14
—, open 15
Sets, distance between 14
Singularity 203
—, essential 224
—, isolated of a multiple character 309
—, —, of a simple character 222
—, of an element 296
Singularities, criterion for locating 300
Sink 45
Sokhotskii's theorem 227
Source 45
Stereographic projection 12
Substitution of one series in another 195
Symmetry transformation 73

Taylor series 137
Theorem of residues 252
— on the composite contour 166, 169
— — — correspondence between boundaries 329
— — — — — — —, converse of 329
— — — decomposition of an analytic function into a power series 175
— — — one-to-one and continuous mappings 23
Transformation of symmetry 73
—, stationary points of 64
Trigonometric functions 91
— —, inverse 125

Uniqueness of expansion into power series 137
— theorem 184
— — of conformal mapping 327

Velocity circulation along the curve 45
— potential 47, 354
—, vortex of 46
Vortex point 355
— source 356

Weierstrass's characteristic of the uniform convergence 138
— formula 242
— theorem on uniformly convergent series of analytic functions 183

Zero of a function 190